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ANNUAL REPORT

OF THE

CHIEF SIGNAL OFFICER OF THE ARMY

TO THE

SECRETARY OF WAR

FOR

THE YEAR 1885.

IN TWO VOLUMES.

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APPENDIX 71.

RECENT ADVANCES
IN
METEOROLOGY,

SYSTEMATICALLY ARRANGED IN THE FORM OF A TEXT-BOOK DESIGNED
FOR USE IN THE SIGNAL SERVICE SCHOOL OF INSTRUCTION AT
FORT MYER, VA., AND ALSO FOR A HAND-BOOK IN THE
OFFICE OF THE CHIEF SIGNAL OFFICER.

PREPARED UNDER THE DIRECTION OF

BRIG. AND BVT. MAJ. GENERAL W. B. HAZEN,
CHIEF SIGNAL OFFICER OF THE ARMY,

BY

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Meteorological Society, of the Royal Meteorological Society, London,
and of the German Meteorological Society.

BY AUTHORITY OF THE SECRETARY OF WAR.

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PREFACE.

In former times meteorology was for the most part merely descriptive, and meteorological work consisted mostly in making and recording observations and taking averages. Little was known with regard to the laws of the phenomena observed, and but little attention given to theoretical research. The most essential parts of meteorology were then comprised in elementary treatises and a few popular and descriptive works on special meteorological subjects. Only a comparatively small amount of reading and study was then required to master everything at hand and to fit any one for further research or to do creditable meteorological work.

It is very different now. During the last quarter of a century there has been, in almost every country, great activity, both in making and recording observations and in theoretical and experimental research. Many very important laws have recently been deduced theoretically and confirmed by observation and experiment, and much which was formerly mysterious is now clear. Solar and terrestrial radiation, the conditions determining temperature, the relation between the amounts of solar heat received on different parts of the earth's surface and the corresponding resulting temperatures, the effect of the deflecting force of the earth's rotation in the mechanics of the atmosphere, the theory of the general motions of the atmosphere, and of cyclones, tornadoes, water-spouts, hail-storms, &c., are subjects which have recently received much attention and are now beginning to be pretty generally understood. Many have taken a part in the recent advancements of meteorology, either directly in doing meteorological work and prosecuting meteorological researches, or indirectly in making physical experiments which have an important bearing upon them. The accumulation of books and papers, therefore, in different countries and in different languages, comprising the results of profound investigations involving the higher principles of physics and complex mathematical analysis, and having an important bearing upon meteorology, has become immense.

To any one who would engage in any line of meteorological work, and especially of original research, a knowledge not only of correct principles and the best methods, but also of what has been already done, is of very great importance. Else there is danger that he will work upon wrong principles and with inferior methods, or else that he will spend much time unnecessarily in doing what has been already well done, or in attempting to discover what has already been discovered. It is lamentable to see often how much time and labor are spent in this way to no useful purpose. The amount of preliminary reading and study, therefore, which is now necessary to qualify one to do creditable meteorological work, or to engage in original research, is very much increased.

But where much has been written upon any subject it must be admitted that there is generally a considerable part which in the end is of little importance, however important it may have been at the time, as a step in the march of progress. The truth has been attained generally little by little, and much that is at first published on any subject is often not only defective but even erroneous. What is most needed is a knowledge of the best principles, methods, and results attained, without regard to the different steps; but in order to be sure of this it has been necessary to read and study all that has been written, and to exercise a good judgment in selecting the most important. This is more than all can afford to do. It is therefore best if this can be done in some measure by one or a few for all.

The object of the following work has been to select from the material on hand some of the more important principles, methods, and results arrived at, mostly during the last quarter of a century, and to present them in the form of a text-book of the higher meteorology, supplementary to more elementary works, for use in the meteorological work of the Signal Service. The subject, however, has become now so vast, embracing so many things of very great interest and importance, that justice cannot be done to it in one volume of moderate size. Much, therefore, of great interest is necessarily either excluded entirely or only briefly referred to. This has especially been done in the case of papers which are well known and accessible. No space could be given to a description and explanation of the many recent inventions and improvements in meteorological instruments for making and recording observations. If the work seems therefore defective it is hoped that the omissions will be found to be in some measure supplied by the numerous references through the work to books and papers from which, in general, further details can be obtained, and likewise much additional and useful matter on the subject under consideration.

It was at first intended to simplify as much as possible the mathematical methods of treatment in the various researches and problems, so as to have them more easily comprehended than those usually adopted in professional papers, and also to give more expansion in the processes; but the want of space has prevented this in a great measure. With this in view, however, methods in some cases have been adopted which, though less elegant than others, it was thought would be more easily understood. This has been especially the case in the formation of the fundamental equations of the general motions of the atmosphere and of cyclones. This, in the more general and more elegant methods of treatment, is considered very difficult, and will be more readily understood by first studying the method here used. It is often well also to have the same subject presented in different shapes. Many of the subjects treated, however, are naturally very complex and difficult, and presented in any shape cannot be comprehended without considerable study.

This work is thought to contain much of a high order, which, though somewhat difficult to understand, is yet of very great importance even in the ordinary meteorological work of the Signal Service. The proper exposure of thermometers requires a knowledge of the laws of radiation and absorption, and the conditions determining temperature; the theory and proper use of the psychrometer, a knowledge of the kinetic theory of gases, and of diffusion and thermal conductivity; and the reductions of barometric observations to sea-level, a knowledge of all the principles involved in hypsometry.

An important feature of the work, it is thought, is the formulæ and tables which are so frequently needed in meteorological computations and discussions of observations, with the examples for their application. Even the samples given of the best method of application of the formula will be found important to many, for, although this is in general no very difficult matter, yet much time is saved in hasty applications of the formulæ, when a sample of the method is given systematically in detail. The book will therefore be very convenient as a hand-book in doing many kinds of meteorological work.

In the prosecution of the work I have to acknowledge the receipt of much valuable aid from Professor Abbe, who took an interest in the work, and furnished me with much reading matter and many references to papers having an important bearing upon the subjects treated in the work.

CHAPTER I.

THE CONSTITUTION AND PHYSICAL PROPERTIES OF THE ATMOSPHERE.

I.—CONSTITUENTS OF THE ATMOSPHERE.

1. The gaseous envelope surrounding the earth called the air, and considered as a whole, the atmosphere, is a mechanical mixture of several simple gases, of which the two principal ones are oxygen and nitrogen. These are very nearly permanent constituents, being subject to only very slight variations in their relative proportions at different times and places. Besides these there is a variable proportion of aqueous vapor, less permanent than the principal constituents, which varies from almost absolute dryness to one-twentieth or more of the whole. There are also measurable quantities of carbonic acid, ozone, and ammonia, and traces of nitric acid, carbureted hydrogen, and in cities, of sulphureted hydrogen and sulphurous acid.

Oxygen and nitrogen.

2. Air which contains only oxygen and nitrogen, having been freed from aqueous vapor and purified of its carbonic acid and other constituents of small proportion, is said to be *pure and dry*. Such air at the earth's surface, according to the numerous analyses of Dumas and Boussingault,[1] was found to contain on the average 20.81 parts of oxygen and 79.19 of nitrogen by volume in 100 parts. These proportions, however, have been somewhat changed in the results of more recent analyses.

From more than 100 analyses of air made by Regnault at Paris in the year 1848[2], the most feeble quantity of oxygen in 100 parts of pure dry air was found to be 20.913, the strongest, 20.999, and the general mean about 20.96. The analyses of air taken from various places around Paris gave about the same results.

From a series of daily analyses in duplicate of air at Hudson, Ohio, continued for six months, Professor Morley[3] has obtained for the ratio of oxygen to the sum of oxygen and nitrogen 20.949 per cent., with a probable error of only .0016 per cent.

The proportion of oxygen, in analyses made at various times and different parts of the earth, does not vary generally more than from 20.9 to 21.0 per cent., but in some cases, mostly in warm climates, the proportion of oxygen is found to be as low as 20.3. Jolly[4] obtained from 45 analyses, made during every month of the years 1875 and 1877, a range from 20.47 to 20.96 per cent. of oxygen. The per cent. was greatest for polar and least for equatorial winds.

---

[1] In the Appendix will be found a list of the authorities referred to in this text.

The air collected by Messrs. Welsh and Green in balloon ascents, made by them under the direction of the Kew Committee of the British Association, and analyzed by Dr. Miller, gave, at the height of 13,460 feet, 20.888 of oxygen; at the height of 18,000 feet, 20.747; and at the height of 18,630 feet, 20.888 per cent.; while air collected at the surface of the earth at the same time contained 20.92 per cent.[5] These results indicate a slight decrease in the proportion of oxygen in the upper strata of the atmosphere.

Carbonic acid.

3. The proportions of carbonic acid which have usually been found in the air range from 3 to 5 parts by volume in 10,000, but recent and more accurate analyses make it a little less. Analyses by MM. A. Müntz and E. Aubin[6] in Paris gave ranges from 3.22 to 4.22 per 10,000 parts in cloudy weather, and in clear weather 2.89 to 3.01; and in the country, from 2.70 to 2.97 by day, and 3.00 for the mean at night. The proportion from analyses generally seems to be a little greater in cities than in the country.

M. I. Reiset, by experiments repeated 193 times, in calm and windy weather, and in the midst of storms, and with air on the sea-shore and in the country, in the woods, and in Paris during the years 1872, 1873, and 1879, has found that the proportion of carbonic acid varies from 2.94 to 3.01 in 10,000 parts, where air is not confined. These results have been verified by M. Franz Schultze at Bostick, who found as a mean, with only slight variations, for 1869, 2.8668; for 1870, 2.9052; and for the first six months of 1871, 3.0126.[7] These differ but little from the proportions obtained from numerous analyses by Mr. G. F. Armstrong, namely, 2.9603 by day, and 3.2999 by night.[8] The proportion at night is usually found to be a little greater than during the day.

Müntz and Aubin have found for the proportion of carbonic acid in the air at the top of the Pic du Midi (2,877 meters) 2.86 in 10,000 volumes.[9] Hence the proportion seems to be about the same at high altitudes as at the earth's surface.

From the daily analyses of air on Montsouris during the years 1876–1881, inclusive, have been obtained the following monthly values in liters for the amount of carbonic acid in 100 cubic meters of air:[10]

January	30.6	August	29.9
February	30.5	September	29.9
March	29.9	October	29.9
April	29.7	November	30.0
May	30.3	December	30.9
June	30.9		
July	30.4	Year	30.2

From these results it does not appear that there is any annual inequality or much change of any sort during the year.

Although the amount of carbonic acid in the air is small, yet it plays an important part in the operations of nature. Animals consume oxygen and exhale carbonic acid as a product of their respiration, and great quantities of it issue from the mouths and fissures of volcanoes. On the other hand, plants consume carbonic acid and give out oxygen, and great amounts of it have been consumed in the formation of carbonates. In former geological periods the amount of carbonic acid was, no doubt, much greater than now, but at present the supply and consumption are most probably about equal, and the amount existing in the air unchanged.

Ammonia.

4. From the same analyses of air on Montsouris have been obtained the following monthly values of the quantity of ammonia in milligrams in 100 cubic meters of air:[10]

January	1.8	August	2.5
February	2.1	September	2.3
March	2.5	October	2.2
April	2.2	November	2.0
May	2.2	December	1.9
June	2.3		
July	2.3	Year	2.2

These values indicate an annual inequality in the amount of ammonia in the air, having its maximum about the warmest part of the year. It seems to vary, however, in different years. There was obtained for 1877, 3.2; for 1878, 1.8; for 1879, 2.1; and for 1880, 1.8.

Ozone.

5. Ozone, since the time of its first discovery by Schönbein, of Basle, in 1840, has usually been regarded as a constituent of the atmosphere, present in small quantities at all times and places. It is an allotropic state of oxygen in the form of transparent gas, one molecule of ozone containing three atoms, while one of oxygen contains only two. Hence its density is 1½ times that of oxygen, or 1.657.

Ozone is produced by passing a stream of electric sparks through a tube in which a current of dry oxygen is passing; by the electrolysis of water; by consuming phosphorus slowly in a globe of moist air; and by the discharge of a Leyden battery. It is supposed to be produced in nature by strokes of lightning through the air; by the slow discharges of electricity by means of trees, towers, and other high objects; by both slow and rapid combustion; and by evaporation.

The third atom in the molecule of ozone is supposed to be loosely combined with the other two, so that it readily enters into combination with other substances when brought into contact with them. Hence its great oxidizing power. It is, therefore, a bleaching and disinfecting agent, and readily discharges the colors from litmus paper and a number of vegetable substances. A blue indigo solution shaken up in air rich with ozone soon loses its color. A proof of the disinfecting properties of ozone is found in the fact that in all places where foul gases are found in great quantities, as in the neighborhood of stables, filthy alleys and sewers, and likewise in the rooms of our dwelling houses, the air is void of ozone. This is because it decomposes, and so is consumed by, these substances. On account of the disinfecting properties of ozone, thunder-storms, which produce it, are thought to be great purifiers of the air.

The usual test of the presence of ozone in the air is based upon the well-known peculiarity of ozone to decompose iodide of potassium. It is upon this property that Schönbein's ozone test papers and ozonometer are based. While these may be regarded as pretty sure tests of the presence of ozone, yet the different shades or degrees of discoloration produced on the paper by the air furnishes only a very imperfect measure of the quantity of ozone, and at best it is only a relative measure. This arises from the impossibility of determining accurately the shade of color in the scale which corresponds to a given proportion of ozone, and also from the fact that the reactions usually ascribed to ozone may be due in part to nitric and other acids in the air. Much also depends upon the eye of the observer, for it is well known that many persons are very defective in the perception of different colors and shades of color. In making such observations it is also necessary to take into account the effect of the wind, since on a windy day a greater effect is produced on the paper, not because there is more ozone in the air, but because more air passes over the paper in the same time. For these reasons some meteorologists regard the numerous ozone observations made since the time of Schönbein as being of little value, and advise their discontinuance,[11] while others have regarded the existence of ozone at all, as a general constituent of the atmosphere, as an open question. It is, however, now rendered pretty certain from chemical processes that there is not only ozone always present in the atmosphere, but likewise that it exists in measurable quantities. Houzeau has estimated that the land air at the height of two meters contains $\frac{1}{450000}$ of its weight of ozone, and Schönbein that it contains $\frac{1}{50000}$ of its weight of ozone after a thunder-storm. From the analyses also of the air at Montsouris the following monthly values of ozone in milligrams for each 100 cubic meters of air have been obtained:

January	1.1	August	1.2
February	1.5	September	1.1
March	1.4	October	1.1
April	1.1	November	0.9
May	1.2	December	0.7
June	1.2		
July	1.3	Year	1.15

The average is about four times greater than Houzeau's estimate, but only about half as much as Schönbein's after a thunder-storm.

Composition of dry air.

6. From what precedes, the average composition of dry air—neglecting the slight and mostly unmeasurable traces of various gases and other elements found in it—may be put as follows: oxygen, 20.95; nitrogen, 79.02; and carbonic acid, 0.03 parts in 100 by volume. The several relative densities are: oxygen, 1.10563; nitrogen, 0.97137; carbonic acid, 1.5201. Hence we have by weight, oxygen, 23.16; nitrogen, 76.77; and carbonic acid, .046 parts in 100. These relations, however, do not quite satisfy the condition that the sum of the volumes multiplied into the densities shall equal the weight of the whole.

II.—PRESSURE AND WEIGHT OF THE ATMOSPHERE.

Absolute pressure.

7. Since the pressure of the atmosphere depends upon the force of gravity, it is necessary to have at the outset a clear conception of force and the measure of force. It will be best, therefore, to consider here some of the elementary principles of force and motion, although this is a subject which belongs more properly elsewhere, and to simplify the matter as much as possible let us consider simply force and motion in one direction, since this will be sufficient for our present purpose. Force acting in a given direction is that which changes or tends to change the momentum of a body moving in that direction. Where the body is entirely free the whole force is spent in increasing its momentum, and hence the proper measure of force is the rate by which the momentum, in this case, is increased. Let

F=the force, acting in the direction s, which changes or tends to change the velocity and momentum in that direction;
m=the mass of the body;
p=the pressure caused by this force when the body is not free;
u=the velocity of motion.

Now, since $m\frac{du}{dt}$ is the rate by which the momentum is changed, we have

$$F=m\frac{du}{dt}=m\frac{d^2s}{dt^2} \tag{1}$$

Hence, we have $Fdt=mdu$, and integrating this for a unit of time (that is, from $t=0$ to $t=1$) we get, regarding F as being constant during the time,

$$F=mu_1 \tag{2}$$

in which u_1 is the velocity generated in a unit of time in the direction of the force. Hence, force is measured by the mass multiplied into the velocity generated in a given direction in a unit of time.

When the body is not free to move at all on account of some resistance, as in the case of a body upon a table, or a column of atmosphere upon the earth's surface, the force causes *pressure*, and we then have $F=p$, for, the whole force being spent in producing pressure, the force becomes equal to the pressure.

But if the body can move in the direction s, but is not entirely free, a part of the force is then spent in giving momentum to the body and the balance in causing pressure, by which the resistance is overcome, and we then have,

$$F=p+m\frac{du}{dt} \tag{3}$$

If the moving body is retarded by friction, as in the case of a body falling through the air, or a part of the air itself moving vertically through the rest, putting

$f=$ the part of the force required to overcome the friction to a unit of mass,

we then have,

$$F=p+m\frac{du}{dt}+fm \tag{4}$$

From the nature of friction a part of the force is communicated to the contiguous part of the air or fluid through which the body moves and adds to the pressure in the direction of motion, and so the pressure of the body is diminished.

8. Where the force F is that of gravity, and consequently in the direction of motion in a vertical direction, putting

$g=$the acceleration of gravity,

we get from (2) and (3) by putting g for u_1 in (2).

$$p=mg-m\frac{du}{dt}-fm \tag{5}$$

in which u is descending velocity. Hence the pressure varies with gravity, and is consequently different at different latitudes and altitudes for the same mass m.

If a body falls through a resisting medium, its velocity is accelerated until the resistance arising, either from the inertia or the friction of the medium passed through, exactly equals the force, after which the velocity becomes uniform and du vanishes, and we then have

$$p=mg-fm \tag{6}$$

9. As the pressure depends upon the acceleration of gravity g, it now becomes necessary to have an expression of g in terms of the latitude and altitude of the body. Let

$g'=$ the value of g at sea-level;
$G=$ the value of g' at the parallel of 45°;
$g'_e=$ the value of g' at the equator;
$\lambda=$ the latitude;
$h=$ the altitude;
$r=$ the earth's mean radius.

We then have at sea-level, from Colonel Clarke's recent determination of the figure of the earth from pendulum experiments (11),

$$g' = g'_{\circ}(1 + .005226 \sin^2 \lambda) = g'_{\circ}(1 + .002613 - .002613 \cos 2\lambda)$$
$$= 1.002613\, g'_{\circ}\left(1 - \frac{.002613}{1.002613} \cos 2\lambda\right) = G\,(1 - .002606 \cos 2\lambda)$$

The numerical coefficient of cos 2λ in this expression is a little greater than those generally used heretofore, but being based upon the most recent researches upon the figure of the earth, in which all previous experiments have had due weight, it must be regarded as the most accurate. It is a little less, however, than the preliminary value obtained by Professor C. S. Peirce.[12]

For any altitude, h, in open space above the earth's surface, since the value of g is inversely as the square of $(r + h)$, the distance from the earth's center, we have

$$g = g' \frac{r^2}{(r + h)^2} = \frac{g'}{1 + \frac{2h}{r}}$$

neglecting very small quantities of the third order in the development in the last form of expression. This expression of g is strictly correct in open space above the earth's surface, and requires a slight modification for the most usual cases where the station of observation is upon a high plateau or mountain top. With the known value of r the numerical coefficient of h in the expression is 0.000000314. The correction, however, introduced by Poisson, which makes this coefficient equal 0.000000194, is now known to be very erroneous. It was made upon the hypothesis that the matter beneath the plateau or mountain top down to the general sea-level is so much additional attractive matter above sea-level; but both from theoretical considerations and from observation it has been shown that this is not the case.[13] It is merely matter raised from beneath and not additional matter, and diminishes only very slightly the decrease in the force of gravity with increase of elevation. Observation, however, shows that there is a slight effect, as we would expect from the displacement of the attractive matter, and we shall therefore put in the expression of g a value of r, represented by r', such as to make the numerical coefficient of h equal 0.0000003. This gives a reduction depending upon h very little greater than those of Professor C. S. Peirce,[12] which differ a little with different forms of mountain or plateau. With the preceding expression of g' that of g above becomes, neglecting insensible quantities of the third order,

(7) $$g = Gn$$

in which, neglecting very small quantities of a third order,

(8) $$n = \frac{1}{(1 + .002606 \cos 2\lambda)\left(1 + \frac{2h}{r'}\right)}$$

With this expression of g (6) gives

$$p = mGn \tag{9}$$

which, by §8, is applicable to solid, and also fluid, bodies at rest.

10. From this and (7) we get for some other mass m',

$$p' = m'Gn' = m'g$$

Hence, if $p' = p$ and $n' = n$, we have $m' = m$—that is, equal pressures correspond to equal masses, and consequently the masses are proportional to the pressures where the force of gravity is the same on both masses. Equality of masses is therefore determined by equality of pressures, and the latter is determined by means of balances. If, however, the masses were not subject to the same force of gravity, as would be the case if one end of the beam of the balances were in high and the other in lower latitudes, or the one mass were suspended from the end of the beam at a higher altitude than the other, then the value of n, by (8), would be different in the two cases, and equality of masses would not be indicated by equality of pressures. In weighing, however, the mass and the counterpoise are both subject to the same force of gravity, and consequently equality of masses is indicated by equality of pressures.

If we now put

σ = the base of a vertical column resting on the earth's surface
ρ = its density,

we shall have, for a column of the height h,

$$m = \sigma\rho h \tag{10}$$

With this (9) becomes

$$p = \sigma\rho h\ Gn \tag{11}$$

The preceding expressions are applicable in the case only of a body of uniform density and of a magnitude for which the value of g is not sensibly different for different parts. For a very high column in which n, by (8), is sensibly different for different altitudes, and in which the density φ also varies with the altitude, (11) is strictly applicable to an infinitely thin stratum only, and hence we shall have in this case

$$dp = Gn\sigma\rho dh \tag{12}$$

We shall have in this case,

$$p = \sigma G \int n\rho dh \tag{13}$$

in which the integration must be from the base to the top of the column. By (8) n in this expression is a known function of h, and it is necessary that ρ, before integration, shall be expressed in some function of h such that the expression is integrable.

If (13) is applied to a vertical column of the atmosphere, p is the pressure of such a column upon the surface σ, which may be at the surface of the earth or at any altitude h' above the surface, the integration commencing at that altitude and extending to the top.

Weight.

11. The *weight of a body* is its pressure relative to that of some assumed standard unit, as a kilogram or a pound, when both, as in the act of weighing, are acted upon by the same force of gravity. If a body presses as much as 100 kilograms subject to the same force of gravity, it is said to weigh 100 kilograms; and this is a measure of the mass of the body. In weighing a body at different latitudes or altitudes we get the same weight, although the pressures differ, for equality of masses depends upon the condition of the equality of the forces acting upon the counterpoising mass and the body weighed, and not upon the absolute amount of this force.

Barometric pressure.

12. In the preceding expressions the pressures are given in terms of the force of gravity as defined in §7, and they consequently depend upon the mass of the body. The pressure of the atmosphere upon the earth's surface, or upon itself at any given altitude h, is measured by the height of the column of mercury of standard temperature in a barometer which, by well-known principles of hydrostatics, has the same pressure as a column of atmosphere extending to the top, and having the same base as the mercurial column, whatever that may be. This measure of the pressure of the atmosphere is, therefore, independent of the base σ in (11), and depends only upon the height and varying density of the column of mercury. If we apply (11) to the mercurial column of a barometer, and put

t=the temperature of the mercury;
Δ=its density;
Δ_0=the value of Δ where $t=0$;
B=the uncorrected height of the mercurial column of the barometer;
β=the coefficient of vertical expansion with increase of t;

we shall have

$$\Delta=\frac{\Delta_0}{1+\beta t} \tag{14}$$

Putting Δ for ρ and B for h in (11) in this case, we get for the pressure of the mercurial column

$$p=\sigma\Delta_0 GB\frac{n}{1+\beta t}=\sigma\Delta_0 GP \tag{15}$$

in which, by means of (8), we have sensibly

$$P=B\frac{n}{1+\beta t}=\frac{B}{(1+.002606\cos 2\lambda)\left(1+2\frac{h}{r'}\right)(1+\beta t)} \tag{16}$$

On the parallel of 45°, and at sea-level, where $\cos 2\lambda=0$ and $h=0$, and for temperature $t=0$, we have $P=B$; and hence P is the height of the mercury at temperature of 0°, when subject to the force of gravity at sea-level on the parallel of 45°, called the *standard* force of gravity.

Since by (15) P is proportional to p, it is a true *relative* measure of it, called *the barometric pressure*, and where σ, G, and Δ_0 are known, the *absolute* pressure, as defined in §7, becomes known from it. But B, the observed height of the mercury, even when corrected for temperature, cannot be used as a true measure of the pressure under all circumstances, since it is not proportional to p, the thing measured, but the relation between it and the pressure changes with both a change of n and t in (15)—that is, with a change of the latitude and altitude, and also with a change of the temperature of the mercury.

The observed height of the mercury B is called the uncorrected height, and corrected for temperature is usually called the barometric pressure, though, as we have seen, it is not, even then, a true measure of the pressure. If in (16) we put for n its value in (8), we get

$$P=B+c \tag{17}$$

in which, neglecting insensible quantities of a third order,

$$c=-B(0.002606\cos 2\lambda+\frac{2h}{r'}+\beta t) \tag{18}$$

is the correction to reduce the observed barometric height to that of standard gravity, and temperature $t=0$. But the correction can in general be more conveniently made by (16) with the use of logarithms, except for sea-level, where $h=0$, and B can in general be regarded as a constant. In this case two small tables can be constructed giving the values of the corrections for given values of the arguments λ and t.

Since the absolute pressure of the mercurial column is equal to that of the atmospheric column of the same base, we have, in a state of static equilibrium, the same value for p in (13) and (15), and hence we get as an expression of the barometric pressure of the atmosphere at the place of observation,

$$P=\frac{1}{\Delta_0}\int n\rho dh \tag{19}$$

13. In the atmosphere or any simple gas ρ is a function of both the pressure and the temperature. Let us put

τ=the temperature of any given mass m of the air or gas.
V=its volume under some assumed standard of pressure P_0.
V_0=the value of V at temperature $\tau=0$.
α=the coefficient of increase of pressure or of volume with increase of τ, in terms of their values at $\tau=0$.

We then have by the law of Boyle, usually called Mariotte's law, the temperature remaining the same.

$$d(PV)=0 \tag{20}$$

for all values of the pressure P.

By the law of Charles and Gay-Lussac we have

$$d(PV)=P_0V_0\alpha d\tau \tag{21}$$

The integration of this from $t=t'$ gives

$$PV=P_0V_0[1+\alpha(\tau-\tau')]$$

If the initial temperature in the integration is $t'=0$, it becomes

$$PV=P_0V_0(1+\alpha\tau) \tag{22}$$

Since the volume is equal to the mass divided by the density, we have $V=m:\rho$, and putting $\rho_0=$ the value of ρ corresponding to $V=V_0$, we consequently have $V_0=m:\rho_0$. With these values of V and V_0, (22) gives

$$\rho=\frac{P}{P_0}\frac{\rho_0}{1+\alpha\tau} \tag{23}$$

14. In a *mixture of gases* of different densities let

$p_1, p_2 \; . \; . \; . \; . \; p_e=$the pressures or tensions at any given temperature τ;

$v_1, v_2 \; . \; . \; . \; . \; v_e=$the corresponding specific volumes—that is, volumes of unit of mass;

$p'_1, p'_2 \; . \; . \; . \; p'_e=$the partial pressures corresponding to the specific volume V_0, which make up the standard pressure P_0 of the mixture;

$v'_1, v'_2 \; . \; . \; . \; v'_e=$the specific volumes under pressure P_0.

We shall thus have by Boyle's law $p_ev_e=P_0V'_e=p'_eV_0$, and hence

$$p'_e=\frac{P_0}{V_0}V'_e \tag{24}$$

The partial pressures are therefore proportional to the partial volumes.

Let us now put

$\delta'_e=$the relative densities corresponding to the volumes v'_e at the temperature $\tau=0$;

$\rho'=$the absolute density of the mixture under pressure P_0 and temperature $\tau=0$;

Since the density of the mixture is equal to the sum of all the weights—that is, sum of all the volumes multiplied into their absolute densities—divided by the sum of the volumes, we shall have

$$\rho'=\frac{\Sigma v'_e\rho_0\delta'_e}{\Sigma v'_e}=\frac{\Sigma v'_e\rho_0\delta'_e}{V_0}$$

And since $\rho_0\delta'_e=$the absolute densities, the sum of the partial volumes must equal the volume V_0 of the whole mixture.

For any pressure P and temperature τ, since the densities are as the pressures and inversely as the absolute temperatures, by (23) we have

$$\rho=\frac{P}{P_0}\frac{\rho'}{1+\alpha\tau}=\frac{P}{P_0}\frac{{}_0\rho}{1+\alpha\tau}\Sigma\frac{v'_e\delta'_e}{V_0} \tag{25}$$

If we put

$p=$the tension of aqueous vapor;

$e=$its tension relative to that of the atmosphere;

$v'_s=$its volume;

We shall have for the relative tension of aqueous vapor

$$(26)\qquad \frac{v'_3}{V_0}=\frac{p}{P}=e$$

Putting v'_2 for the volume of carbonic acid, we have by §6

$$\frac{v'_2}{V_0}=.0003$$

We shall then have, for the volume of pure and dry air,

$$v'_1=V_0-v'_2-v'_3$$

We have for the relative densities $\delta'_1=1$, $\delta'_2=1.529$, and $\delta'_3=0.622$ With these densities (25) gives

$$\rho=\frac{P}{P_0}\frac{\rho_0}{1+\alpha\tau}\left(\frac{(V_0-v'_2-v'_3)+1.529\ v_2+0.622\ v'_3}{V_0}\right)$$

This, by means of the preceding values of $v'_s : V_0$, may be reduced to the following form:

$$(27)\qquad \rho=\frac{P}{P_0}\ \frac{\rho'_0}{1+\alpha\tau}(1-.378\ e)$$

in which

$$(27')\qquad \rho'_0=\rho_0\left(1+\frac{.529\ v'_1}{V_0}\right)=1.00016\ \rho_0$$

is the density of dry air with the average amount of carbonic acid. Hence the density of dry air is increased a little by the carbonic acid, but by (27) it is decreased by the aqueous vapor. For pure air we have $v_1=0$, and then ρ'_0 becomes ρ_0.

From (19) we now get, by means of (8) and (27), neglecting very small insensible quantities of the third order of $0.378\,e$ and of $\frac{2h}{r'}$,

$$(28)\qquad \frac{dP}{P}=d\ \log\ P=-\frac{dh}{lq\left(1+\alpha\tau+0.378\,e+\frac{2h}{r'}\right)}$$

in which

$$(28')\qquad q=1+0.002606\cos 2\lambda \qquad\qquad l=\frac{P_0\Delta_0}{\rho'_0}$$

The last member is negative, since P increases as h decreases, P here denoting the pressure of the air above the upper station.

If we put

l=the height of a column of gas of uniform density, which has the same weight as the mercurial column of the same base and the standard height P_0;

δ=the relative density of the gas under pressure P_0 at temperature $\tau=0$;

We shall then have, since the height of the column of gas and uniform density and P_0, the height of the mercurial column, are inversely as the densities,

$$(29)\qquad l=\frac{P_0\Delta_0}{\rho_0\delta}$$

In the case of an atmosphere of pure dry air δ becomes unity.

The value of l in (28′) is *the height of a homogeneous atmosphere of dry air*, since ρ'_0 is the average density of such air. Although the mercury and air have the same *pressure*, yet, by our definition in § 11, they have not the same *weight*, since different parts of this column are acted upon by forces of gravity which vary slightly with the height, while that acting upon the mercury is sensibly the same for all parts. The height of a homogeneous column of air of the same *weight* or mass would be a little less.

Since in (28′) ρ'_0 and P_0 are proportional, whatever may be the assumed standard of pressure P_0, it is evident that the value of l is constant, so that the height of a homogeneous atmosphere, so called, is the same, whatever may be the altitude of the base from which it is reckoned, and is consequently the same if reckoned from the top of a very high mountain or from sea-level. Since ρ'_0 becomes ρ_0 in the case of pure air, it is seen from (28′) that the value of l is slightly greater in this case.

15. The next step now is to integrate (28). In order to this it is necessary to assume that τ and e are certain functions of h. If we assume that τ and e increase in proportion to the altitude, we can put

$$\tau=f(h+a) \qquad\qquad e=f'(h+a')$$

in which f and f' are the rates of increase of τ and e with reference to h, and in which a and a' are certain constants such as to make τ and e equal τ' and e' when h becomes h'. With these values of τ and e we get as a function of h

$$(30) \qquad \varphi(h)=1+\alpha f(h+a)+.378f'(h+a')+\frac{2h}{r}=1+\alpha\tau+0.378e+\frac{2h}{r}$$

From this we get

$$d\varphi(h)=c\,dh$$

in which

$$(31) \qquad c=\alpha f+0.378f'+\frac{2}{r}$$

Hence we get from (27) in Naperian logarithms

$$d\log P=\frac{q}{lc}\cdot\frac{d\varphi(h)}{\varphi(h)}=\frac{q}{lc}d\log\varphi(h)$$

The integration of this gives

$$(32) \qquad \log\frac{P'}{P}=\frac{q}{lc}\cdot\log\frac{\varphi(h)}{\varphi(h')}$$

in which P' is the value of P corresponding to $h=h'$, and in which

$$(33) \qquad \varphi(h')=1+\alpha f(h'+a)+0.378f'(h'+a')+\frac{2h'}{r}=1+\alpha\tau'+0.378e+\frac{2h'}{r}$$

Let us now put

$$m=\tfrac{1}{2}\varphi(h)+\tfrac{1}{2}\varphi(h') \qquad\qquad n=\tfrac{1}{2}\varphi(h)-\tfrac{1}{2}\varphi(h')$$

We shall then have $m+n=\varphi(h)$ and $m-n=\varphi(h')$, and hence (32) becomes

$$(34) \qquad \log\frac{P'}{P}=\frac{1}{lqc}\log\frac{m+n}{m-n}$$

By a known development of Naperian logarithms we get in common logarithms

$$\log(m+n)=\log m+M\left(\frac{n}{m}-\frac{n^2}{2m^2}+\frac{n^3}{3m^3}-\&c.\right)$$

$$\log(m-n)=\log m+M\left(-\frac{n}{m}-\frac{n^2}{2m^2}-\frac{n^3}{3m^3}-\&c.\right)$$

in which M is the modulus of common logarithms. Hence we have

$$(35)\quad \log\frac{m+n}{m-n}=\log(m+n)-\log(m-n)=2M\left(\frac{n}{m}+\frac{n^3}{3m^3}+\&c.\right)$$

From the preceding expressions of m and n we get, by means of (30), (33), and (31),

$$(36)\ m=1+\tfrac{1}{2}\alpha(\tau'+\tau)+0.189(e'+e)+\frac{h+h'}{r'}\qquad\qquad n=\tfrac{1}{2}c(h-h')$$

From these expressions of m and n it is seen that the second and following terms in (35) are extremely small and insensible terms in comparison with the first, and may be neglected. Retaining the first term only in the last number of (35), we get from (34), by means of (35) and (36),

$$(37)\qquad \log\frac{P'}{P}=\frac{M}{lq}\,\frac{(h-h')}{1+\tfrac{1}{2}\alpha(\tau'+\tau)+0.189(e'+e)+\dfrac{h+h'}{r'}}\ .$$

The preceding method of integration, though giving the same result, is more satisfactory than that of more simple ones depending upon a development of the last number of (28), in which quantities of the second and lower orders of αt, &c., are neglected in the integration. In the preceding development of (35) the effect of the neglected terms is readily seen to be of no importance in any case.

Constants of experiment and observation.

16. In the preceding expressions there are certain constants, entering either directly or indirectly, which have to be determined from observation and experiment, and these expressions cannot be used for any practical purposes until the values of these constants are determined and substituted in them. These constants are G, r, α, ρ_0', $\varDelta_0$, and l.

The value of the constant G by the latest determination is 9.806056.

The value of r, the mean radius of the earth, according to the recent researches of Colonel Clarke,[11] is 6,367,323^{m}, but only a rough approximate value is needed.

The value of α has been determined by Regnault in two ways. From (22) we get for constant volume,

$$P-P_0=\alpha P_0\tau$$

and for constant pressure,

$$V-V_0=\alpha V_0\tau$$

The initial pressure P_0 at temperature $\tau=0$ being known, with the observed increase of pressure $(P-P_0)$ corresponding to any increase of temperature τ, the volume remaining the same, the first of these gives α. In like manner the observed increase of volume $(V-V_0)$, corresponding to any increase of temperature τ, the *pressure* now remaining constant, the last of these also gives α. The value obtained for α by Regnault,[14] from numerous experiments for ordinary pressures and temperatures of the atmosphere, was 0.003665 by the former method, and 0.003670 by the latter. The value given by both methods would be the same if the laws of Boyle and Charles were strictly correct. The latter, which is the coefficient of dilitation under constant pressure, is the one which properly belongs to and is usually adopted in the preceding expressions. The value of ρ_0', or ρ_0 in the case of pure air, is the density of such air at the assumed standard pressure of $P_0=760^{mm}$ at temperature $\tau=0$, when the mercury is acted upon by the force of gravity on the parallel of 45° and at sea-level. At Paris, where we have, in (8), $\lambda=48^\circ\ 50.2'$ and $h=60^m$, the value found by Regnault for the density of pure and dry air at temperature $\tau=0$, and under pressure of 760^{mm}, after being corrected for some small errors of calculation, is $\rho=0.00129321$, the density of pure water at the temperature of maximum 4° *C* being the assumed unit of density. This, however, is not the density of such air under a pressure of 760^{mm} of the mercury acted upon by standard gravity, but that at Paris, which is greater. Hence this density must be reduced to that of 760^{mm} of the mercury, subject to the standard force of gravity.

Since by (27) the density of the air is as the pressure, and this by (7) and (11) varies as g, which on the parallel of 45° and at sea-level is G, we have by means of the ratio of $g:G$ given in (41), and the observed value of ρ given above,

$$(38)\qquad \rho_0=\frac{0.00129321}{[1-.002606\cos(97^\circ\ 40.4')]\ [1-.0000003\times 60]}=0.00129278$$

With this value of ρ_0 we get from (27′)

$$(39)\qquad \rho'_0=1.00016\times 0.00129278=0.00129298$$

for the density of dry air with the average amount of carbonic acid in it. With this special value of density (27) becomes a general expression of the density of the air under all conditions of pressure P, temperature τ, and relative tension of aqueous vapor e.

The value of Δ_0 usually adopted is 13.59593, which is the value determined from experiment by Regnault.

With this value of Δ_0 and the value of ρ_0 above, and the standard value of $P_0=.76^m$, we get from $(28'_2)$ $l=7991.7^m$ as the height of a homogeneous atmosphere of ordinary dry air. With the value of ρ_0 in (38), we get $l=7993.0$ for pure dry air.

We have, therefore, for the constants in the preceding expressions, depending either directly or indirectly upon experiment and observation,

$$(40)\qquad \left\{\begin{array}{l} G=9.806056^{m}=32.1727 \text{ feet} \\ r=6367323^{m}=20890548 \text{ feet} \\ r'=6364463^{m} \\ l=7991.7^{m}=26220 \text{ feet} \\ \alpha=0.003670 \\ \rho'_0=0.00129298 \\ \Delta_0=13.59593 \end{array}\right.$$

Practical formulæ.

17. By means of these values of the constants, we get from preceding expressions the following formulæ, convenient for practical application:

From (7) and (8), by means of (40_3), we get for the reduction of gravity to the parallel of 45° and sea-level

$$(41)\qquad G=g(1+0.002606 \cos 2\lambda)(1+0.0000003h)$$

If h is given in feet, the coefficient of h is .000000091.

From (27), (39), and (40_4) we get for the general expression of the density of ordinary air, in all the varying conditions of pressure, temperature, and amount of aqueous vapor,

$$(42)\qquad \rho=\frac{P}{P_0}\,\frac{0.00129298}{(1+0.003670\tau)(1+.378e')}$$

Since in the average state of the atmosphere the relative tension e of the aqueous vapor is somewhat in proportion to the temperature, if we put $e=0.289\tau$, the preceding expression becomes

$$(42)'\qquad \rho=\frac{P}{P}\,\frac{0.00129298}{1+0.004\tau}$$

This corresponds very nearly with the change in the coefficient of τ above, made by Laplace, in order to take into account approximately the effect of aqueous vapor on the average. Of course in special and extreme cases it may be in error to the full amount of the whole change in the coefficient. At the temperature $\tau=0$, the whole effect of the aqueous vapor is lost, and below that temperature the correction for this effect has the wrong sign; but at these low temperatures, the whole effect of the aqueous vapor upon the density is scarcely sensible. If, however, (42)′ were used for very low temperatures the error would be considerable.

Where the temperature is given in degrees Fahrenheit (42) becomes

$$(43)\qquad \rho=\frac{P}{P_0}\,\frac{0.00129298}{[1+0.002039(\tau+32^\circ)]\,[1+0.378e]}$$

Since only the relative values of p to P and of P to P_0 in (26) and (27) are used, the barometric pressures and the tensions of vapor may be expressed in either millimeters or inches, or the absolute pressure may be used. For degrees Fahrenheit the denominator $1+0.004\tau$ in (42)′ becomes $1+0.002222(\tau-32^\circ)$.

The expression of (27) is applicable to any simple gas by putting $e=0$, and using the value of ρ_0 for that gas. It therefore becomes in this way applicable to aqueous vapor, in which case P becomes the tension of the vapor, p. The density with relation to pure dry air in this case being 0.622, relatively to water it will be $0.00129278 \times 0.622 = .0008041$, and hence we shall have

$$(44) \qquad \rho=\frac{p}{P_0}\,\frac{.0008041}{1+0.003670\tau}$$

Since the densities ρ_0 in the preceding expressions are those of standard pressure P_0, the values of P_0 and p must be the observed values corrected by (18).

If we put

v=the volume of a given quantity of air or any gas;
m=its mass or weight;

we shall have

$$(45) \qquad m=\rho v$$

As our density unit here is that of pure water at standard temperature, m is the mass relatively to that of an equal volume of such water, or, in other words, it is the weight expressed in unit volumes of water. In order, therefore, to have the weight of gas in other units, as grams or grains, we must multiply the result given by (45) into the number of grams or grains in a unit volume of standard water. The gram being a cubic centimeter of such water, if the assumed unit volume of air is a cubic meter, as is most usual, then the unit volume of water weighs 1,000 kilograms, and hence in this case the result of (45) must be multiplied into 1,000 to give the weight or mass in kilograms. We shall therefore have in kilograms, if v is given in cubic meters,

$$(46) \qquad m=1{,}000\rho v$$

The weight of one cubic meter of air or any gas is therefore $1{,}000\rho$, the value of ρ being determined for any special gas under any given conditions of pressure, temperature, &c., by the preceding formula. For the weight of a cubic meter of pure dry air, therefore, under standard pressure and temperature, using for ρ in this case the value of ρ_0 in (38), we get $m=1.29278$ grams. With the usual amount of carbonic acid it would be by (45) a little greater.

If the assumed unit of volume is one cubic foot, and we wish the weight of any given volume v of air or any gas in grains Troy, we shall then have

$$(47) \qquad m=436845\rho v$$

since 436845 is the number of grains Troy in a cubic foot of water at standard temperature of 4° C., instead of the English standard 62° F., the densities of ρ given above being those with reference to water of the former standard of temperature. Hence, the weight of a cubic foot of dry and pure air under standard temperature and pressure, using for ρ in (47) the value of ρ_0 in (38), is

$$m=436845 \times 0.00129278=564.75 \text{ grains.}$$

18. With the value of $M=0.434295$ we get from (37) by means of $(28'_1)$ and (40)

$$(48)\quad \log\frac{P'}{P}=\frac{(1-0.002606\cos 2\lambda)\,(h-h')}{18401.6\left[1+0.001835(\tau'+\tau')+0.189(e'+e)+\dfrac{h'+h}{6604463}\right]}$$

in which h must be expressed in meters. The denominator in this expression can be put without sensible error into the form

$$[1+0.00183(\tau'+\tau)]\,[1+0.189(e'+e)]\,[1+0.0000003(h'+h)]$$

and with this the preceding expression can be put into the following form:

$$\begin{aligned}(49)\qquad \log(\log P'-\log P)=\log H+\log(1-0.002606\cos 2\lambda)-4.264855\\ -\log[1+0.001835(\tau'+\tau)]-\log[1+0.189\\ (e+e')]-\log[1+.0000003(2h'+H)]\end{aligned}$$

in which H is put for $(h-h')$.

These accurate expressions are sometimes needed in the reverse problem of barometric hypsometry and reductions of the barometer to sea-level, but in most cases it is sufficient to include the vapor term in that of the temperature, as in $(42)'$, rejecting the very small term depending upon $(h'+h)$, and then (49) becomes

$$\begin{aligned}(50)\qquad \log(\log P'-P)=\log(1-0.002606\cos 2\lambda)+\log H-4.264855\\ -\log[1+0.002(\tau'+\tau)]\end{aligned}$$

In the preceding expressions $4.264855+\log[1+0.00183(\tau'+\tau)]$ is given by Table I by deducting 0.51493, $\log(1-002606\cos 2\lambda)$ from Table VI, by taking the complement, and $\log(1+.0000003\,(2h'+H)$, by Tables IV and V. Also $\log[1+.002(\tau'+\tau)]$ is given by Tables I and IV very nearly by deducting 0.51493.

When H is expressed in English feet the constant logarithm 0.00106 must be deducted from Table I. This is because the table is adapted to the case in which the barometric pressure B, uncorrected for diminution of gravity with altitude, is used, whereas in these formulæ P the true pressure enters instead of B.

Since the ratio only between P' and P in the preceding expressions is used, it is not necessary that the corrected values should be used except so far as they depend upon the temperature of the mercury and the difference of elevation, in (32), since the part of the correction depending upon latitude is the same at both stations. It is also immaterial whether the barometric pressures P' and P are expressed in millimeters or inches, since the ratio between the two is the same in both cases.

If H is given in feet, the constant logarithm in these expressions is 4.780848.

Pressures on a unit of surface.

19. In a cubic unit of pure water we have in (11) ς, ρ, and h, each equal to unity, and hence the pressure of such a unit is Gn. But since n, by (8), is a function of latitude and altitude, this is not a constant

for all localities, and hence cannot be used as an absolute measure of pressure. If, however, we take the pressure of such a unit on the parallel of 45° and at sea-level, then n becomes unity, and the pressure of such a unit becomes G. If the unit is a cubic decimeter or liter, it is called a kilogram. A kilogram, therefore, has different pressures at different latitudes and altitudes, and in order to become an absolute measure of pressure we must have the pressure of the kilogram at some standard locality, for which the parallel of 45° and sea-level is usually assumed. We get thus from (11) for the standard pressure in kilograms

(51) $$p=\sigma\rho hn$$

For the pressure of a column on a unit of surface we must put $\sigma=1$. In the case of the mercurial column we have $\rho=\varDelta$ and $h=P$, and hence we get for the pressure in kilograms on a square decimeter

(52) $$p=\varDelta Pn$$

Putting p_0=the value of p corresponding to $P=P_0$, we get at sea-level on the parallel of 45°, where $n=1$,

(53) $$p_0=\varDelta_0 P_0=13.596\times 7.6=103.33 \text{ kilograms}$$

for the standard pressure of the mercury, or of the counterpoising column of the atmosphere, on a square decimeter of the earth's surface. This standard pressure being that of a column of mercury of the height of 760^{mm} at temperature $\tau=0$, we must use $\varDelta_0$, the density of the mercury at that temperature.

In English measures we have for the pressure in pounds avoirdupois upon a square inch (Appendix),

(54) $$p_0=103.33\times\frac{2.20485}{(3.9371)^2}=14.697 \text{ pounds}$$

since 1 kilogram=2.20485 pounds and 1 decimeter=3.9371 inches.

To obtain the absolute pressures for other latitudes and altitudes we must multiply these results into n, and hence the pressure, measured in units of the standard pressure of the kilogram or the pound—that is, its pressure on the parallel of 45° and at sea-level—vary both with latitude and altitude.

20. If we put v=the value of V in (22) belonging to a kilogram of airs, we shall have in this case, from (22) and (52),

(55) $$pv=p_0v_0(1+\alpha\tau)=p'v'$$

if we put $p'v'$ for the value of $p_0v_0(1+\alpha\tau)$ when $\tau=\tau'$. If we now assume the meter as the unit of measure instead of the decimeter, we have from (53) $p_0=10333$ kilograms, and v_0 equal the volume of a kilogram of air in cubic meters under the standard pressure p_0, and hence v_0 equal to unity divided by the weight of a cubic meter of air, or

(56) $$v_0=\frac{1}{1000\times .00129298}=.77341$$

using the value of ρ'_0 in (39) for the density or weight of a cubic decimeter of dry air. This is the volume of dry air with the average amount of carbonic acid. This for dry and pure air would be a little greater, since in that case we would have ρ_0 in (38) instead of ρ'_0 in (39).

In the case of moist air the density by (27) is decreased in the ratio of 1 to $(1-.378e)$, and consequently the volume is increased very nearly as 1 to $(1+.378e)$. Hence we have in this case

$$v_0=0.77341(1+e) \tag{56'}$$

With this value of v_0 and the value above $p_0=10333$ we have from (55)

$$pv=RT \tag{57}$$

in which

$$\left\{\begin{aligned} &R=\frac{p_0v_0(1+.378e)}{a}=\frac{10333\times0.77341(1+.378e)}{273}=29.274(1+.378e)\\ &T=a+\tau=273^\circ+\tau\\ &a=\frac{1}{\alpha}=273\text{, very nearly}\end{aligned}\right. \tag{58}$$

With $t=-a$ in (62) we should have $pv=0$, and hence $p=0$, since v could not become 0 unless we suppose the atoms and molecules to be merely mathematical points. When p vanishes, all motions of the molecules, and consequently heat and temperature, by the kinetic theory of gases, must vanish. Hence $-a=-273^\circ$ is called the zero of absolute temperature and T above becomes the absolute temperature reckoned from this zero. This, however, is strictly correct only upon the hypothesis that the air is a *perfect gas*—that is, a gas in which the laws of Boyle and Charles hold strictly for all ranges of temperature and pressure—which, according to experiment, is not the case.

From (57) we get the relation between the tension or pressure p on a unit of surface, and the volume v of a kilogram of air, and the absolute temperature T.

Deviations from the laws of Boyle and Mariotte.

21. The law of Boyle and Mariotte requires that (20) should be satisfied in any change of pressure and volume where the temperature remains the same. But Regnault[14] and others have shown by experiment that not only in carbonic acid and other gases which are transformed into liquids under considerable pressure at ordinary temperatures, but likewise in atmospheric air and nitrogen, $d(PV)<0$ for a range of pressures from one to thirty atmospheres, but that in the case of hydrogen it is the reverse. In the former the volume decreases and the density increases, with increase of pressure in a ratio greater than that required by Boyle's law, but in hydrogen in a less ratio.

The results of Regnault's experiments with atmospheric air are represented by the following formula:[15]

$$\frac{PV}{P_0V_0}=1-.00110535\left(\frac{V_0}{V}-1\right)+.0000193809\left(\frac{V_0}{V}-1\right)^2 \tag{59}$$

If we confine ourselves to small ranges of P and V, the last term in this expression is so small that it may be neglected in comparison with the second, and then from the relation $V=m:\varphi$ and $V_0:V=P:P_0$, the latter, though not strictly correct where Boyle's law is not, being sufficiently so in a very small term, we get from (59)

$$\rho=\frac{\rho_0 P}{P_0\left(1-.00110535\frac{P-P_0}{P_0}\right)}=\frac{\rho_0 P}{P_0}\left(1+.00110535\frac{P-P_0}{P_0}\right)$$

The effect on the density ρ of the deviation from Boyle's law is expressed by the last term in the last form of expression, and it is seen that it is extremely small for all ordinary ranges of atmospheric pressure, even where P is not more than $\frac{1}{2}P_0$, supposing that the law of deviation, which has been deduced from pressures greater than P_0, can be extended down to pressures much below P_0. The effect, therefore, of this deviation upon all the formulæ in which the density of the air ρ enters is extremely small and of no consequence whatever.

22. The experiments of Regnault from which (59) has been deduced extended from one to twenty atmospheres of pressure, and so did not include pressures below one atmosphere. It is, therefore, not safe perhaps to extend it to pressures which are much less than an atmosphere or to those much above twenty atmospheres, for any empirical formula devised to represent the results of experiment cannot be safely extended far beyond either limit of the range of the experiments.

From (59) we get by differentiation

$$\frac{d(PV)}{P_0}=.00110535\frac{V^2_0}{V^2}dV-.0000387618\left(\frac{V_0}{V}-1\right)\frac{V^3_0}{V_2}dV$$

In order to satisfy Boyle's law we must have by (20) the last number of this equation equal 0. We therefore get, when this law is satisfied,

$$\frac{V_0}{V}=\frac{P}{P_0}=1+\frac{.00110535}{.0000387618}=29.52$$

This gives nearly thirty atmospheres for the value of P where this condition is satisfied. For smaller values of P this gives $d(PV)<0$, but for greater ones $d(PV)>0$. Hence in the former case the volumes decrease, and consequently the densities increase, in a ratio greater than that required by Boyle's law, within the range of pressures to which (59), from which the preceding relation is deduced, can be applied.

By (59) we, of course, have $PV=P_0V_0$ for the initial change of P from P_0, but this condition is also satisfied with a value of $V_0:V$ or $P:P_0=$ 60 very nearly, and hence with a value of P equal to about 60 atmospheres of pressure. For greater pressures than this, if (59) can be extended so far, we always have $PV>P_0V_0$. This, however, has been shown to be the case for high pressures from numerous and more recent experiments made through a range of pressures extending even up to those of many atmospheres, not only for the atmosphere, but also other gases. According to Amagat the change in the atmosphere from $PV<P_0V_0$ to $PV>P_0V_0$ occurs at a pressure of about 65 meters or about 86 atmospheres, which is considerably greater than that obtained above by the extension of (59) to high pressures. In the case of nitrogen under a pressure of 430 atmospheres Amagat[16] found PV to be about one-fourth greater than P_0V_0.

The greatest uncertainty in the deviations from Boyle's law is in the pressures below an atmosphere, and especially where these pressures become very small. This is because very small errors and inaccuracies in the experiments and the measurements give rise to errors in the results which are large in proportion to the whole. Perhaps the most accurate researches on the Boyle-Mariotté law, at least for small pressures, are those of Mendeleef.[17] Where the deviations are such as to make, as the pressure is increased, $(dPV)>0$, he very appropriately calls them *positive* deviations, and where $(\alpha PV)<0$, *negative* deviations. In all the gases under small pressures he found positive deviations, and the deviation in the case of hydrogen was five times as great under a pressure of 120^{mm} as it was under a pressure of 400^{mm}. The deviations in this gas are positive throughout from the lowest to the highest pressures. There is consequently no pressure where it follows Boyle's law rigorously. In atmospheric air, for all pressures under 600^{mm}, he found positive deviations, and for pressures greater than an atmosphere its deviations became negative, but for pressures above 100 atmospheres the compressibility became positive again. Hence there are two points at which the compressibility of the air conforms to Boyle's law, the one somewhere between a pressure of 600^{mm} and an atmosphere, and the other somewhere below 100 atmospheres. This, we have seen, according to the results of Regnault's experiments, represented by (59), would be at a pressure of about 30 atmospheres. But from the law of (59), extended back to low pressures, we nowhere get positive deviations, and therefore it does not represent Mendeleef's results obtained for low pressures. But this, for reasons already given, could scarcely be expected. According to Mendeleef the compressibility of the lower half of the atmosphere must be very nearly that of Boyle's law, and if so, δP very nearly vanishes and becomes entirely insensible to observation. It has generally been supposed that the rarer the air or any gas is, either from diminished pressure or increased temperature, the more nearly its compressibility conforms to Boyle's law, but this is not in accordance with Mendeleef's researches.

Deviation from Charles's law.

23. This law, usually called the law of Gay-Lussac, requires that α in (21) should be the same at all pressures and temperatures whether P or V is the variable. Experiments have not been made with a range of temperature sufficiently great to determine whether α is different at different temperatures, but the value of α is found to be greater for constant pressure than for constant volume, and also that it increases in both cases with increase of pressure. The following are the coefficients for constant volume obtained by Regnault in his researches upon the

subject for air at the several pressures at 0° and 100° corresponding to the relative densities in the first column:[15]

Relative density.	Pressure at 0°.	Pressure at 100°.	α
	mm.	mm.	
0.1444	109.72	149.31	0.0036482
0.3501	266.06	395.07	.0036542
0.4987	375.23	510.97	.0036572
1.0000	760.00	....	.0036650
2.2084	1678.40	2286.09	.0036760
2.8218	2144.18	2924.04	.0036894
4.1800	3655.56	4992.09	.0037091

In other gases the values of α differed more from one another as the pressure is greater. In carbonic acid, for instance, it changed from .0036856 to .0038593 for a change of pressure from 758.47^{mm} to 3589.07^{mm}. Denoting the coefficient of expansion with constant pressure by α', he obtained for a pressure of 760^{mm} $\alpha'=.0036706$, and for a pressure of 2525^{mm} $\alpha'=.0036944$. For the first Mendeleef obtained $\alpha'=.003683$.[17]

The laws of Boyle and of Charles are deducible from the kinetic theories of heat and of gases, but for some reason there are disturbances causing slight deviations from these laws. The value of α, therefore, is not strictly a constant in different gases or at different pressures in the same gas. With these values (58_3) gives different values of a, or place of the zero of absolute temperature with relation to the temperature of freezing. It becomes a question, therefore, which of these values we should use to get the most probable value of a, or place of absolute zero. The values of α above obtained by Regnault differ considerably within a comparatively small range of pressure in the experiments, and there is, therefore, a considerable uncertainty with regard to the true place of the absolute zero.

Examples.

24. (*a*) The relative density of hydrogen being .0692, what is the height of a homogeneous atmosphere of hydrogen?

From (29), with the known densities of air and mercury, we have in this case

$$l=\frac{.76\times 13.596}{.00129278}=115503^{m}$$

(*b*) If the barometric pressure at the earth's surface is 745^{m} and temperature $\tau'=22.5°$, what is the pressure at the altitude of $10,000^{m}$, on the parallel of 45°, supposing the temperature to decrease at the rate of 0.5° for each 100^{m} of altitude, and the air to have the average amount of moisture?

In this example we have

$$\tau=22.5^{\circ}-0.5\times100=-27^{\circ}.5$$

and hence

$$\tau'+\tau=-5^{\circ} \text{ and } 1+.002('\tau+\tau)=.99$$

From (50) we then have

$$\log(\log P'-\log P)=4-4.26485-(.99564-1)=-.26049$$

and then

$$\log P=\log P'-.54892=2.87216-.54892=2.32324$$

and

$$P=210.5^{mm}.$$

(*c*) The barometric pressure at sea-level being 30 in. and the temperature 56° F., with average amount of moisture, what is the barometric pressure at the height of 4 miles, supposing the temperature to decrease at the rate of 1° F. for each 300 feet of altitude?

In this example use (50) with constant log 4.780848, and express $\tau'+\tau$ in degrees centigrade.

(*d*) In the second example what is the relative density of the air at the height of 10,000^{m} (6.2 miles)?

In this example use (42)' for the absolute density and divide by $\rho_0=$ .00129278, the density of pure dry air.

(*e*) What is the weight in grains of the aqueous vapor in a cubic foot of air at temperature of 80° F. if the tension of the vapor is 0.75 in.? (44) and (47), putting $P_0=29.92$ inches, and expressing the temperature in centigrade degrees.

(*f*) What is the acceleration of gravity g at sea-level at the north pole? What on top of Pike's Peak, latitude 38° 50′, altitude 14,134 feet? (41) and (40).

III.—DIFFUSION AND ARRANGEMENT OF THE CONSTITUENTS.

Dalton's theory.

25. More than half a century ago Dalton,[18] starting with the suggestion of Newton that the law of the expansive power of a gas is explained upon the hypothesis that each molecule repels every other adjacent one with a force which is inversely as the distance, and assuming that, in a mixture of gases, this repelling force exists only between the molecules of the same gas, established the principle that in a mixture of gases each one, whatever the initial state of the mixture may be, assumes after a time the same status which it would have in case the others were not present. And upon this principle, of course, the several constituents of the atmosphere, acted upon by the force of gravity, would each form a separate atmosphere, entirely independent of the others, and the relations between the pressure, altitude, and temperature would be that given in (37). Hence the different constituents were regarded as vacuums to one another; and meteorologists formerly separated the aqueous atmosphere from the dry, and gave the pressure of each sepa-

rately. Although by this theory each one of the constituents is finally diffused through the whole space, as it would be if the others were not present, yet this does not take place almost instantaneously, as would be the case, unless the space were very large, if they were as vacuums to one another, but it is proved by experiment that this takes place very slowly. In the case, therefore, of the aqueous vapor, which is being continually formed by evaporation and condensed back again into water at different rates at different times and places, such an atmosphere cannot be formed; and the distribution of the vapor is found to be very irregular, varying greatly at different places on the earth's surface not far from one another, and decreasing with increase of altitude at varying rates, and on the average with a rate greater than that of an independent atmosphere, since the evaporation is mostly at the surface and the vapor is slowly diffused upward. The observation, therefore, of any local tension of aqueous vapor, gives us only the amount of this vapor in that locality, and leads to great error if we suppose that this is the tension of a regular atmosphere of vapor.

From hygrometric observations made by different observers at various places, on the Himalayas, Mount Ararat, Teneriffe, the Alps, and also in the balloon ascensions of Welsh & Glaisher, Dr. Hann[24] has deduced the following empirical formula showing the relation between the vapor tension p' at sea-level or some lower station, and p the vapor tension at a station at the altitude h above the lower station.

$$(60) \qquad p=p'e^{-\frac{h}{6517}}=p'10^{-\frac{h}{2830}}$$

in which p and p' are measured by millimeters of mercury, and h is expressed in meters. This of course is a kind of average relation for all times and places, and does not apply with much accuracy often in individual cases.

Diffusion according to the kinetic theory of gases.

26. According to this theory the expansive force and tension of a gas arises from the constant motion and encounters of the molecules flying with great velocity and, on account of the numerous encounters and deflections, in all directions. If the gas is more dense in any region than in neighboring regions, the outward expansive force from this region is greater than the counteracting force toward it, and more molecules pass out than into it, until uniformity of density is established, or, where there is an external force acting in one direction, until an arrangement takes place that makes the tension of all parts exactly equal to the pressure arising from the external force.[19]

Where there is a mixture of two or more gases in the same region or inclosure unevenly distributed in the different parts, the effect of these numerous encounters is evidently to equalize the distribution and make the densities of the different parts of each gas and the relative proportions of the several gases the same throughout, where there is no external force, as gravity, acting upon the mixture. And even where there is such a force, as in the case of the atmosphere, it has been shown in the analytical researches of Maxwell[20] and Stefan[21] that the final arrangement of each gas, where the mixture is quiescent, is exactly that which would take place if the others were not present, and hence the final status of the gas in all cases is the same as that deduced from Dalton's theory.

Although the final state of the mixture is the same in all cases by both theories, yet the rates by which this state is brought about are very different. According to Dalton's hypothesis there is no resistance to the diffusive force of the several gases except the inertia of the molecules moved, while according to the kinetic theory of gases, the expansion and diffusion of each gas are resisted and retarded by the encounters with the molecules of all the other gases.

According to Stefan the rate of interdiffusion of two gases in a tube must satisfy the following equation:

$$\frac{dp_1}{dt}=k\frac{d^2p_1}{dx^2}$$

in which p_1 is the partial pressure of either gas, x is distance in the direction of diffusion in the tube, and k is a constant, called the diffusive constant, which is inversely as the total pressure p, supposed to be the same at all times in different parts of the tube, and directly as the square of the absolute temperature. It also depends upon the nature of the two gases and upon the product of their densities. Where the gas is evenly distributed the second member vanishes, and consequent by p_1 becomes a constant both with regard to space and time, and this is the final state to which the gas must tend.

Arrangement of the constituents of the atmosphere.

27 According to the theory of diffusion of gases there is an even distribution laterally of the several constituents of the atmosphere where there are no disturbing causes, but vertically the relations of partial pressure and altitude must be those which would take place in the case of a simple gas, as given in (37) in the case of the atmosphere regarded as such. This equation is therefore applicable to each of the several atmospheres of oxygen, nitrogen, and carbonic acid, and also to aqueous vapor where the temperature and pressure are not such as to condense any part of it into water, provided we use the value of l deduced from (29) with the value of the density ρ_0, and for P_0 the partial pressure belonging to the several gases.

With the partial volumes per cent., given in § 6, we get from (24) for the partial pressures in millimeters corresponding to the several atmospheres of oxygen, nitrogen, &c.,

$$p_s'=\frac{760}{100}v_s'$$

since the partial volumes there given are in terms of $V_0=100$.

If we put p'_o, p'_n, p'_c for the partial pressures of oxygen, nitrogen, and carbonic acid which make up the dry air pressure P_0, and v'_o. v'_n, and v'_c the corresponding partial volumes which make up the volume V_0, we have from § 6, $v'_o=20.95$, $v'_n=79.02$, and $v'_c=0.03$. With these values the preceding expression gives $p'_o=159.22$, $p'_n=600.55$, and $p'_c=0.23$. These values of p' take the place of P_0 in all the preceding formula for the atmosphere, or the place of P' in (37), since the pressure at the base is P_0 in this case, when we apply this formula to the separate component atmospheres.

The partial pressures, which by (24) are proportional to the partial volumes, had usually been taken proportional to the partial weights until the error was pointed out by Dr. Hann,[23] and this error, even yet, is sometimes left uncorrected.

In applying (37) to each separate atmosphere of the component gases, we must in each case use the value of l given by (29) with the value of δ belonging to that gas, and hence it requires the second member of (37) to be multiplied into δ, where the constant $M:l$ is determined for an atmosphere of dry air. We shall, therefore, have for each atmosphere

$$(61) \qquad \log\frac{p'}{p}=\delta\log\frac{P_0}{P}$$

Having computed log (log P_0—log P) by (49) or (50) for the whole atmosphere, it is then only necessary to add log δ to get the logarithm of the first member of the preceding equation. Or, if we wish to compute only one of the partial pressures for different altitudes, we must in each case correct the constant logarithm, 4.264855, by adding log δ. The values of δ to be used in the cases of oxygen, nitrogen, and carbonic acid are given in § 6. With these values and the values of p'_o, p'_n, and p'_c above, the values of p_o, p_n, and p_c in the following table have been computed from (61) for the parallel of 45°, after the values of P were first computed from (49) for the different altitudes H, h' here being the value of h corresponding to $P_0=760^{mm}$, and for the assumed temperatures in the second column, which are so taken as to represent approximately the average decrease of temperature with increase of altitude. To these are also added two columns of the pressures p_v and p_h of an assumed aqueous vapor and a hydrogen atmosphere, with tensions of p'_v and p'_h of 10^{mm} each at the base, computed in the same way.

H'	τ	P	p_o	p_n	p_c	Σp	$\Sigma p-P$	p_v	p_h	Vapor tension of saturation.	By (60)
km.	°	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0	20	760.00	159.22	600.55	0.23	760.00	+0.00	10.00	10 00	17.86	17.86
5	0	419.56	81.68	334.10	0.08	415.86	.30	6.87	9.60	4.57	2.98
10	− 20	217.46	39.94	178.12	0.03	218.09	.63	4.59	9.17	0.94	0.51
15	− 40	108.32	18.48	90.50	0.01	108.99	.67	2.98	8.74	0.15	0.09
20	− 60	51.05	8.06	43.60	......	91.66	.61	1.86	8.30	....	....
30	−100	9.25	1.23	8.27	......	9.50	.25	0.64	7.37	....	....
40	−140	1.24	0.14	1.15	......	1.29	.05	0.18	6.42	....	....
50	−180	0.11	0.01	0.10	......	0.11	.00	0.04	5.42	....	....
60	−220	0.00	0.00	0.00	......	0.00	.60	0.00	4.41	....	....

It is seen from this table of results that the pressure, and consequently the density, of the atmosphere and of each one of its constituents diminish rapidly with increase of altitude, so that at the height of about 50 kilometers (31 miles) the pressure of the whole atmosphere above is only 0.11^{mm}, and at the height of 60 kilometers it may be regarded as a vacuum. The greater the relative density of the constituent the more rapidly the pressure diminishes, so that in the upper regions the relative densities of the several constituents become different from what they are below. At the earth's surface the densities of the oxygen and nitrogen, since they are as their pressures, are as 1 to 3.774, while at the height of 20 kilometers they are as 1 to 5.41. The relative amounts of the several constituents of oxygen, nitrogen, and carbonic acid, therefore, in a quiescent atmosphere would be very different at great altitudes from what they are at the earth's surface; but on account of the constant agitation and mixing together of the lower and upper strata of the atmosphere by its general circulation, and by storms and other causes of inversions of the strata, the constituents are never allowed to assume the status which they would by Dalton's theory if the atmosphere were perfectly quiet. The analyses of air, therefore, from great altitudes, indicate only a slightly greater ratio of nitrogen to oxygen than that found at the earth's surface.

The differences between P and Σp arise from regarding the atmosphere as a simple gas in the application of (49) in the computation of the pressures at the different altitudes. These differences indicate the amount of error which would arise from this hypothesis in barometric hypsometry and reductions to sea-level if the atmosphere were quiescent.

Vapor atmosphere.

28. On account of the smaller relative density of aqueous vapor the pressure of its atmosphere, as is seen from the column in the preceding table headed p_v, es donot diminish in proportion so rapidly with increase of altitude as that of the air so long as the former follows the laws of Boyle and Charles, and the temperature and pressure are not such that a part of it is condensed into water. In oxygen and nitrogen there is no point throughout the whole range of pressure from the least to the greatest where, at the ordinary temperatures of the earth's surface, or even at very much lower temperatures, any part of these gases is liquified, and consequently they follow somewhat closely the laws of Boyle and Charles except for very great pressures, where they become much less compressible than a gas which would conform to these laws. But in aqueous vapor the temperature, called *the critical point*, at which it is condensed at some point in the whole range of pressure, is very high, so that some part of it is condensed at ordinary temperatures even with very low pressures.

There is a certain relation between the temperature and the tension of the vapor at which condensation commences which is entirely independent of all other circumstances, and is the same whether the vapor exists alone or is diffused through other gases. The relation can only be determined by experiment, and this has been very accurately done by both Regnault[23] and Magnus with sensibly the same results. The tension of the vapor at which it begins to be condensed at any given temperature, and which is its maximum tension for that temperature, was found by Regnault[23] to be so nearly the same when the vapor existed alone and when diffused with other gases, that he concluded the difference was due to small inaccuracies in the experiments.

It was formerly thought that the diffusion of the vapor through the atmosphere was of the nature of an absorption, and that the maximum tension at any given temperature indicated the greatest amount of the vapor which the air would absorb at that temperature. The air was then said to be *saturated.* But as this is the same where the air or any other gas is not present, the relation between the temperature and tension does not depend upon absorption. Nevertheless, the term *saturation* is still used to express that state of the atmosphere in which it has the greatest quantity of aqueous vapor which it can have at the existing temperature.

Various formulæ have been devised to represent empirically the relation between the temperature τ and the maximum tension of the vapor, e, as determined from experiment for different temperatures, and to serve as an interpolation formula for intervening temperatures. Of these the one most convenient in practical application is the following devised by Magnus, containing only two constants:

$$(62) \qquad e = 4.525 \times 10^{\frac{a\tau}{b+\tau}}$$

in which $a=7.4475$, and $b=234.69$. This represents accurately the relations between temperature and maximum tension as obtained by Magnus from experiment, and very nearly those of Table X, obtained from Regnault's experiments by the International Bureau of Weights and Measures.

From this table are taken the maximum tensions of vapor in the last column but one of the preceding table, corresponding with the temperatures given in the second column. By comparing these tensions with those of the column headed p_v, it is seen that the tensions which could actually exist in a quiescent atmosphere with the assumed temperatures of the second column diminish much more rapidly with increase of altitude than they would if the vapor conformed at all temperatures with the law of Boyle and Charles, in which case the vapor tensions with a tension of only 10 millimeters at the earth's surface, and one much less than the tension of saturation there with the assumed temperature, would be as given in that column. At an altitude considerably less than 5 kilometers the maximum tension becomes less than that of an atmosphere with a tension of 10 millimeters at the base and conforming to the laws of Boyle and Charles, and at an altitude of 10 kilometers, with a temperature of -20°, it is only one-fifth as much. If the dry atmosphere, therefore, were absent or were as a vacuum to the vapor, a vapor atmosphere with temperature at different altitudes as assumed could not exist with the relations between pressure and altitude as given by (49), and by the column headed p_v in the preceding table for the special case of a tension of 10 millimeters at the base. For, according to the last column but one of that table, the vapor above at a moderate altitude would be condensed and the tension reduced very low, and then the vapor below would rapidly ascend on account of the diminished pressure, and in turn be condensed and fall back, so that the actual amount and rate of diminution of tension with increase of altitude, after the vapor had arrived to the altitude of temperature of condensation, would be very nearly that of the last column but one in the preceding table. On account of the diminished pressure at the base and the consequent increased rate of evaporation, the ascent of vapor, its condensation above, and its falling back again in turn to the earth's surface, would be much more rapid than it is in the ordinary atmosphere. But even with the increased rate of evaporation the tension of vapor at the earth's surface and in the lower strata could hardly approximate very nearly to that of the last column but one, usually called the tension of saturation. The tensions in this column of course depend very much upon the assumed temperatures of the preceding table.

In the last column, it is seen, the tensions fall below those of the preceding one, as they must, since only with almost infinite diffusibility could they become as great.

Supposed hydrogen atmosphere.

29. It is interesting and important in this connection to consider what would be the arrangement of an exceedingly rare constituent of the atmosphere, such as hydrogen, and its relation to the other constituents. The relation which would exist between pressure and altitude for the assumed temperatures in the case of hydrogen, regarded as an independent atmosphere and with a tension of 10 millimeters at the base, is seen in the column headed p_h in the preceding table. On account of its small relative density the rate of decrease of pressure with increase of altitude is comparatively slow, so that at the altitude of 50 kilometers, where the pressure of the atmosphere, though 760 millimeters at its base, is very nearly insensible, that of the hydrogen has not been decreased one-half. Though the tension of such a constituent at the earth's surface were so small as to be scarcely observable, yet at the altitude of 40 kilometers it would be the principal constituent, and at 50 kilometers and upwards it would be apparently the only constituent of the atmosphere. Where the relative densities of the constituents, regarded as independent atmospheres, would be very nearly the same at the different altitudes, as in the case of oxygen and nitrogen, a constant agitation of the whole may keep them so mixed up that the proportions

are nearly the same at all altitudes; but in the case of a very rare constituent the tendency would be for it to rise up so far above all the others that it could not become mixed up with them except in the lower strata, and at a considerable altitude it would be the only sensible constituent unaffected by the agitation of the comparatively much denser constituents, which would exist sensibly only lower down near the earth's surface. This is possibly the case with the sun's atmosphere, which, so far as it can be observed, seems to be mostly if not entirely hydrogen, but far down beneath this there may be an atmosphere composed of much denser gases.

IV.—DYNAMIC HEATING AND COOLING OF THE AIR.

30. It is well known that if air is suddenly compressed there is an increase of temperature, and if it expands from its elastic tension the temperature is decreased. The former is shown by the sudden condensation of air in a condensing syringe or pneumatic tinder-box, by which light and dry substances are ignited from the increase of temperature, and the latter is seen in the rarefaction and expansion of air in an air-pump, by which it is soon so cooled as to condense the vapor contained in it. In the vertical motions of the atmosphere arising from various causes the same part of it comes under greater or less pressures according as the motion is a descending or ascending one. As the pressure of the air is increased in descending, its temperature is increased, and as the pressure is decreased and the air expands, the temperature is diminished. In the dynamics of the atmosphere, therefore, it is very important that the law or rate of increase or decrease of temperature with change of altitude and pressure, and the various effects or consequences arising from it, should be well understood.

Relation between work and change of volume.

31. The overcoming of resistance of any kind to motion by a force is called *work*. The *measure* of work is the product of the force into the space through which it acts. If we put

W=the work performed;
F=the force exerted;
s=the space through which the force is exerted;

we have, if the force remains constant through the space s,

$$W=Fs$$

But as the force F generally changes as s does, this is applicable generally to only an infinitely small space ds, so that we shall then have

$$dW=Fds \tag{63}$$

in which F must be regarded as some function, known or unknown, of s.

In the usual expansions and compressions of air arising from meteorological causes, the resistances to be overcome are, in the former case, the external pressure of the atmosphere on all sides and the inertia of the parts moved, and in the latter, the elastic force of the air compressed and the inertia of the part moved. But the inertia, in such expansions and compressions, is usually so small in comparison with the pressure, or the counteracting elastic force, that it may be neglected, and the pressure taken as the force exerted in both cases in the direction of motion.

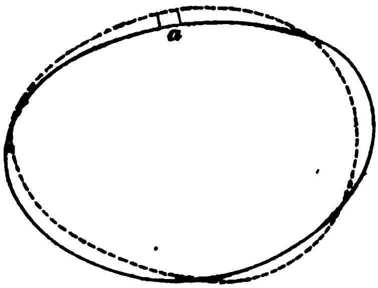


FIG. 1.

Let the solid line in the annexed figure represent a section of the surface of a portion of air of any figure before expansion or compression, and the dotted line the same after an infinitely small change of volume dv from expansion or compression, and let $d\sigma$ represent an infinitely small part of the surface σ at the point a. Letting ds represent the infinitely small space through which the surface at a is moved, we shall have $dv = d\sigma ds$ for the infinitely small change of volume belonging to the infinitely small portion of surface $d\sigma$, and for the whole surface σ.

$$dv = \int_{\sigma} d\sigma ds \tag{64}$$

in which the integral must be taken over the whole surface σ.

If we put dw for the work done by an infinitely small part $d\sigma$ of the surface σ we get from (63), putting in this case the pressure of the atmosphere p on a unit of surface for the force F,

$$dw = pd\sigma ds$$

By integrating over the whole surface, regarding p as a constant, as it is sensibly unless the whole volume of air is very great, we get

$$dW = p\int_{\sigma} d\sigma ds \tag{65}$$

Multiplying both members of (64) by p, we get, by means of this equation,

$$dW = pdv \tag{66}$$

Hence, where the pressure remains the same the increment of work is proportional to the increment of volume. In the case of compression dv is negative, since then the working force, equal to p, is exerted in the contrary direction, and must be taken with a contrary sign.

In the special simple case in which the expanding air is so confined that it can expand, as represented in the annexed figure, in one direction only, the comprehension of the relation of (66) becomes very easy. Let the solid line represent the surface before expansion in this case, and the dotted line the surface after an infinitely small change of volume $dv = \sigma ds$, ds representing the infinitely small distance between the full and the dotted line. The force in this case being the pressure p on a unit of surface multiplied into the whole surface σ, or $p\sigma$, we have the infinitely small increment of work $dW = p\sigma ds$. But σds is the corresponding increment of volume dv. Hence we get, as before, the expression of (66).

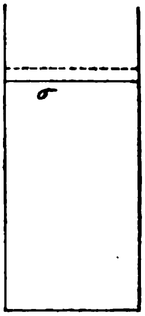


FIG. 2.

Relation between changes of temperature and pressure.

32. According to the mechanical theory of heat, wherever work is done by heat a certain amount of work corresponds to a given amount of heat, the one being always proportional to the other. The usually assumed *unit of heat* in this relation is the amount of heat required to raise a kilogram of water from the temperature of 0° to 1° C. The *unit of work*, called the kilogram-meter, is the work done in raising one kilogram through a height of one meter. But since the kilogram has different pressures on different latitudes and at different altitudes, §10, and work is the overcoming of absolute force, in order to have an absolute unit, which does not vary under different circumstances, and by which, therefore, we can express the true measure of work, we must use some standard pressure of the kilogram, as that on the parallel of 45° and at sea-level.

The number of units of work corresponding to a unit of heat, called *the mechanical equivalent of heat*, can only be determined by experiments. The results obtained by different experimenters in different ways differ considerably, so that there is still some uncertainty with regard to the true value of this constant. The work equivalent obtained by Joule is 772 foot-pounds, in which the unit of heat is the amount of heat required to raise the temperature of a pound of water from 32° to 33° F. This corresponds very nearly with 424 kilogram-meters.

According to more recent experiments of Professor Rowland,[43] in which the temperatures of the mercury thermometers were reduced to those of the air thermometer, or, rather, to the absolute temperatures which would have been given by a perfect gas, this constant, when reduced back to the temperature from 0 to 1°, as required by the definition of a heat unit, is about 431 kilogram-meters. In these experiments the constant decreased with increase of temperature of the water, indicating that the specific heat of water, contrary to the determination of Regnault, at first decreases with increase of temperature up to at least 30°. This result has since been confirmed by the experiments of F.

Neesen,[44] of Berlin, and G. A. Liebig,[45] so that it now seems evident that the results obtained at ordinary working temperatures require to be increased to reduce them to the standard temperature. Joule's result, with this reduction and a correction for the reduction of his temperatures to those of the air thermometer, as determined by Professor Rowland from a comparison of their thermometers, would be only about $\frac{1}{430}$ less than that of Professor Rowland. The value, therefore, of 430 kilogram-meters, which cannot be much in error, or at least must be more nearly correct than the value 424, which has mostly been used heretofore, will be adopted here.

The *heat equivalent of work* is the amount of heat required to do one unit of work, and consequently it is represented by the reciprocal of the mechanical equivalent of heat, or $\frac{1}{430}$, using the corrected value of the mechanical equivalent of heat.

33. The *specific heat*, or *relative capacity for heat*, of air is the amount of heat, expressed in units as above, required to raise the temperature of one unit by weight of air, as a kilogram or a pound, one degree. This, for constant pressure, according to Regnault, is 0.2375. As this, in the case of water, is one unit by definition, it requires less than one-fourth the amount of heat to raise the temperature of a unit by weight of air which it does to raise the same unit of water, through a given number of degrees.

Where the air *expands* as it is heated, it requires a greater quantity of heat to raise its temperature through one degree, since in this case, by (66), work is done, and it is done at the expense of the heat supplied. An additional amount of heat, therefore, the equivalent of the work done, has to be supplied, in order to raise the temperature as much as in the case where the volume is constant, and the whole effect is in the increased tension and temperature of the air. If the tension and volume are both increased, a part of the heat is spent in increasing the tension and temperature and the other in doing work. Let

C = the specific heat of dry pure air when the pressure is constant;
C_v = the same when the volume remains constant;
dQ = an infinitely small increment of heat imparted to a unit of weight of air, corresponding to a change of temperature $d\tau$;
A = the heat equivalent of work, or $\frac{1}{430}$.

The increment of work done by expansion (66) is pdv, and as A is the amount of heat required to do a unit of work, $Apdv$ is the increment of heat spent in doing the increment of work pdv. C_v being the quantity of heat required to raise a unit of weight of air through one degree of temperature without expansion, the increment of heat required to raise its temperature through $d\tau$ is $C_v d\tau$. Hence, as the heat imparted is supposed to be all used in heating the air and doing the work of expansion, we have

$$dQ=C_v d\tau+Apdv \tag{67}$$

From (57) we get by differentiation,

$$pdv+vdp=Rd\tau \tag{68}$$

The difference between C and C_v is equal to the heat used in doing the work of expansion in raising the temperature of a unit by weight of air one degree where the pressure is constant. Putting $dp=0$ in the preceding equation, as we must where the pressure is constant, we then get from this and (66) $dW=Rd\tau$. Integrating from $t=0$ to $t=1°$, we get for the work done in raising the temperature through one degree $W=R$. Consequently the heat required to do this work is AR, since it requires A, or $\frac{1}{430}$ of a unit of heat, to do a unit of work. We therefore get

$$C_v=C-AR \tag{69}$$

With this value of C_v, and the value of $pdv=Rd\tau-vdp$, deduced from (68), (67) gives

$$dQ=Cd\tau-Avdp$$

Dividing by $Apv=ART$ by (57), this equation gives

$$\frac{dp}{p}=\frac{C}{ART}d\tau-\frac{dQ}{ART} \tag{70}$$

In case no heat is imparted from external sources during expansion or compression, or spent, except in doing the work of expansion, we have $dQ=0$, and the last term in (70) vanishes. We then get by integration, bearing in mind that $d\tau=dT$

$$\log\frac{p}{p'}=\varepsilon\log\frac{T}{T'}, \text{ or } \frac{p}{p'}=\left(\frac{T}{T'}\right)^{\varepsilon}=\left(\frac{273+\tau}{273+\tau'}\right)^{\varepsilon} \tag{71}$$

in which p' is the value of p corresponding to $T=T'$, and in which

$$\varepsilon=\frac{C}{AR} \tag{72}$$

Equation (71) expresses the well-known relation, first given by Poisson[25], between the change of pressure and the corresponding change of temperature in the case of dry air, when no heat meanwhile is received from, or imparted to, surrounding bodies or media. It is applicable, therefore, in all cases of sudden expansions or compressions.

Since the specific heat of aqueous vapor, according to Regnault[14], is 0.4805, and consequently considerably greater than that of dry air, the specific heat of the mixture of dry air and the vapor differs somewhat from that of dry air.

If e is the relative tension (26) of the aqueous vapor in the dry air, the weights of dry air and vapor in a kilogram of air are $(1-0.622e)$ of the former and $0.622e$ of the latter. Hence we have for the general expression of the specific heat with constant pressure, applicable to either dry or moist air,

$$C=0.2375(1-.622e)+0.4805\times.622e=0.2375+0.1512e \tag{73}$$

In the case of dry air the terms depending upon e in (58) and (73) vanish, and we get in this case, putting for A its value $\frac{1}{430}$,

$$\varepsilon=\frac{430\times0.2375}{29.274}=3.489 \tag{75}$$

Examples.

1. If dry air at the earth's surface where the pressure is p' and temperature $\tau'=27°$ has rapidly ascended to the altitude where the pressure is $\frac{1}{2}p'$, no heat meanwhile being received, or lost except in the work of expansion, what is its temperature?

In this example $T'=300°$. We therefore have from (71)

$$\log T=\frac{\log\frac{1}{2}}{3.489}+\log 300=2.3908$$

Hence $T=246.0°$ or $\tau=246.0-273=-27.0°$. The decrease of temperature in ascending is $\tau'-\tau=27+27.0=54.0°$.

2. With the same value of τ', the initial temperature, what is the temperature if the pressure is suddenly doubled?

3. If a pressure of 10 atmospheres should be suddenly applied in the compression of air under standard pressure and at a temperature of 60° F., what would be the temperature of the compressed air?

In this example $\frac{5}{9}(60-32)=15.56°$ C. and $T=273+15.56=288.56°$. Hence we have

$$\log T=\frac{\log 10}{3.489}+\log 288.56=\frac{1}{3.489}+2.46025=2.7468$$

Whence $T=558.2°$ C., and $\tau=558.2-273=285.2°$ C.$=545.4°$ F.

Relation between change of altitude and temperature.

34. As any part of the atmosphere ascends or descends it comes under a different pressure. It is therefore expanded and cooled while ascending and the contrary while descending. It consequently becomes important to know at what rate its temperature is changed with change of altitude, not only in the case of dry air, but especially in the case of ascending moist and saturated air. In the latter case the moisture of the ascending current is gradually condensed, and the latent heat given out by condensation can be treated as so much heat imparted

from surrounding sources in the case of dry air, and therefore as being represented by Q in (70). If we put

r=the latent heat of aqueous vapor at temperature τ;
q=the vapor in a kilogram or the specific volume v of air;
dq=the infinitely small amount condensed while the temperature changes by $d\tau$;

we shall have for the heat supplied by condensation,

$$dQ=rdq \tag{75}$$

in which, according to Regnault, we can put with sufficient accuracy

$$r=606.5-0.695\tau. \tag{76}$$

neglecting the terms of a lower order, since they have no weight in short ranges of τ.

The value of q in terms of the tension of the aqueous vapor represented here by e, multiplying the volume into the density, is

$$q=\frac{.622e}{P} \tag{77}$$

By differentiation we get

$$dq=q\frac{de}{e}-q\frac{dP}{P} \tag{78}$$

With this value of dq, (75) gives

$$dQ=rq\frac{de}{e}-rq\frac{dP}{P} \tag{79}$$

and by means of this we get from (70)

$$\frac{dP}{P}=\frac{C}{ART}d\tau-\frac{1}{ART}\left(rq\frac{de}{e}-rq\frac{dP}{P}\right)$$

or

$$\left(ART+rq\right)\frac{dP}{P}=\left(C+rq\frac{1}{e}\frac{de}{d\tau}\right)d\tau \tag{80}$$

From (28), (28′), and (8) we get for moist air,

$$-\frac{dP}{P}=\frac{\rho'_0}{P_0\Delta_0}\frac{ndh}{(1+\alpha\tau)(1+.378e)}$$

Putting for ρ'_0 its equal, $1 : v_0$, and $P_0\Delta_0=p_0$ by (53), this becomes by means of (58)

$$-\frac{dP}{P}=\frac{n}{RT}dh \tag{81}$$

By means of this (80) is reduced to

$$-\left(A+\frac{rq}{RT}\right)ndh=\left(C+rq\frac{1}{e}\frac{de}{d\tau}\right)d\tau \tag{82}$$

From this we get

$$\frac{d\tau}{dh}=-\frac{A+\frac{rq}{RT}}{C+rq\frac{1}{e}\frac{de}{d\tau}}n \tag{83}$$

The last number of this equation expresses the ratio between the change of temperature and the change of altitude in rapidly ascending currents of saturated moist air. As it is negative, the temperature decreases as the altitude increases. As n is a factor, this ratio, by (8), is a little greater in the polar than in the equatorial regions of the earth.

In the practical application of (82) we still need expressions of $\frac{1}{e}\frac{de}{d\tau}$ and of rq convenient in computation. The former of these is obtained most readily from the empirical formula of Magnus (62), for expressing the maximum tension of vapor in a function of the temperature τ The differential of this gives*

$$(84)\qquad \frac{1}{e}\frac{de}{d\tau}=\frac{1}{M}\frac{ab}{(b+\tau)^2}\qquad \left[\log\frac{ab}{M}=3.60472\right]$$

From (76) and (77) we get

$$(85)\qquad rq=(606.5-0.695\tau)\frac{.622e}{P}=(378-0.432\tau)\frac{e}{P}$$

Where vapor is changed directly to snow or ice, as in the case of temperature below freezing, it is necessary to include in rq the latent heat of fusion as well as that of condensation, and this is done by adding to r in (76) the increase of temperature corresponding to this heat, which, as determined by Regnault in the case of snow, is 79.25°. With this increase of the value of r we get in this case, instead of the preceding,

$$rq=(427-0.432\tau)E$$

The vapor tension e is obtained from (62) or more conveniently from Table X. With this thus obtained, and the given barometric pressure P, (26) gives e, and then with this (58) and (73) give the values of R and C to be used in (83) to obtain $D_h\tau$. In this way the value of 100 $D_h\tau$—that is, the change of temperature for each 100 meters in Table XIII has been computed for each of the given values of P in the first column, and temperatures τ at the head of the other columns. The part depending upon E in (58) and (73) is so small that the constants $R=29.272$ and $C=0.2375$ can be used without much error.

From (77), with the values of e taken from Table X, the values of q, the weight of vapor in a kilogram of saturated air, are computed and given in the table for the same arguments P and t.

From (44) and (46) is computed the weight of vapor in a cubic meter of air, and given in the same table for those arguments.

36. Where there is no *condensation*, as in the case of dry air, and

---

*We have in Naperian logarithms, in which system $\log 10=\frac{1}{M}$, $\log e=\log 4.525+\frac{a\tau}{b+\tau}\frac{1}{M}$. Hence $d\log e$, or $\frac{de}{e}=\frac{1}{M}d\frac{a\tau}{b+\tau}=\frac{1}{M}\frac{ab}{(b+\tau)^2}d\tau$, and dividing by $d\tau$, $\frac{1}{e}\frac{de}{d\tau}=\frac{1}{M}\frac{ab}{(b+\tau)^2}$.

even in that of moist air if it is not saturated, and in all cases of descending currents, we have $dq=0$, and hence from (78) and (81) we get

$$\frac{de}{e}=\frac{dP}{P}=-\frac{n}{RT}dh$$

With this value (82) gives

$$-An\frac{dh}{d\tau}-\frac{rq}{RT}n\frac{dh}{d\tau}=C-\frac{rq}{RT}n\frac{dh}{d\tau}$$

or

$$\frac{d\tau}{dh}=-\frac{A}{C}n \tag{86}$$

In *dry air*, C becomes by (73), since e in this case vanishes, equal to 0.2375, and hence we get in this case

$$\frac{dt}{dh}=-\frac{n}{430\times 0.2375}=-0.00979n \tag{87}$$

On the parallel of 45° and at sea-level, where, by (8), $n=1$, we have in a rapidly ascending or descending current of dry air a change of temperature of 0.979° for each 100 meters of change of altitude, the temperature decreasing in ascending and increasing in descending currents. This in English measures is 0.537° for each 100 feet.

In moist but *unsaturated* air the rate is slightly less, since in this case we would have in (86) C, by (73), a little greater than 0.2375.

On account of the factor n, the rates given by both (83) and (86) are a little greater in the polar than in the equatorial regions, though the difference is very small.

All the preceding relations into which A enters, which is the heat equivalent of work, depend upon the dynamic heating and cooling of the atmosphere in its contractions and expansions.

Examples.

1. Compute by (83), in the manner described, the value of 100 $D_h t$ at the parallel of 45° for the pressure $P=600^{mm}$ and temperature $t=15°$.

In this example we get by (62), or from Table X, $e=12.68^{mm}$. With this value and the value of $P=600^{mm}$, we get from (85) $rq=7.814$. Also from (26) and (58) we get $R=29.508$, and from (26) and (73) $C=0.2407$. From (84) we get $\frac{1}{e}\frac{de}{dt}=.06455$. With $T=273°+15=278$, and the value of R above, we get $RT=8499$. With these values and the value of $A=\frac{1}{430}$, (83) gives 100 $D_h t=-0.438°$.

2. In the same example what is the weight of vapor in a kilogram of the saturated air? (77.)

3. What also is the weight of vapor in a cubic meter of the saturated air? (44) and (46.)

Relation between changes of altitude and of density.

37. Putting in (23) for $P : P_0$ its equal $p : p_0$, and for ρ_0, its equal $1 : v_0$, we get by means of (58), for dry air,

$$\rho=\frac{p}{RT}$$

Taking the differential of both numbers, remembering that $dT=d\tau$, we get

$$d\rho=\frac{dp}{RT}-\frac{p d\tau}{RT^2}=\frac{p}{RT}\left(\frac{dp}{p}-\frac{d\tau}{T}\right)$$

Putting for $dp : p=dP : P$ its value in (81), and dividing both members by dh, we get

$$\frac{d\rho}{dh}=-\frac{p}{RT^2}\left(\frac{n}{R}+\frac{d\tau}{dh}\right) \tag{88}$$

With the value of $D_h\tau$ in (83), in the last member of this equation, we have, in the case of rapid vertical currents, the ratio of the increase of density $d\rho$ to that of the increase of altitude dh in a function of the temperature and pressure and known constants, since by (62), (77), and (83) $D_h\tau$ is such a function. Having, therefore, the temperature and pressure, we know the value of the first member of (88). The value of $D_h\tau$ to be used in (88) can be obtained approximately from Table XIII for any given pressure and temperature.

Since n is a factor in the expression of $D_h\tau$, (83), it is a common factor in both terms of (88), and consequently the value of $D_h\rho$ is a little greater toward the poles than it is toward the equator.

In the case of unsaturated air, and therefore in the case of all descending currents, the value of $D_h\tau$ in (86) must be used in the second member of (88). We get therefore in this case

$$\frac{d\rho}{dh}=-\frac{p}{RT^2}\left(\frac{1}{R}-\frac{A}{C}\right)n \tag{89}$$

Since $1 : R$ is larger than $A : C$, the value of $D_h\rho$ is always negative, and hence in the case of rapidly ascending or descending currents the density decreases as h increases, and *vice versa*.

If in the case of a stagnant atmosphere near the earth's surface we have, in (88),

$$\frac{d\tau}{dh}=-\frac{n}{R} \tag{90}$$

and consequently $d\rho=0$, we then have the same density at all altitudes. To satisfy this condition in the case of dry air on the parallel of 45° we must have

$$\frac{d\tau}{dh}=-\frac{1}{29.274}=-.0342^\circ$$

that is, the temperature must decrease 3.42° for each 100 meters of increase of altitude. For moist air and for other latitudes, the value of $D_h\tau$ would be very nearly the same, since the values of n and R in (90) vary very little.

In all the preceding expressions A enters, either directly or indirectly, and consequently they are affected by the dynamic heating and cooling of the atmosphere.

Stable and unstable equilibrium.

38. If when a portion of the atmosphere in a state of static equilibrium receives from any slight temporary cause of disturbance an upward or downward motion, the changed conditions by such movement tend to bring back again the air to its original position, the air is then said to be in a state of *stable equilibrium*, since immediately after such disturbance it is soon restored to its original status. But if, after any such disturbance, the changed conditions tend to continue the initial motion in the direction started, either up or down, the air is then said to be in a state of *unstable equilibrium*, since, although in equilibrium in a quiescent state, yet the slightest disturbance introduces an initial change in the conditions which tends to continue the motion and also to increase the tendency to keep up the motion, until by a complete inversion of the strata, or from some other cause, the conditions are so changed that a stable equilibrium is brought about.

It is well known that if any portion of the air, or of any liquid, or a solid body immersed in a liquid, has a density less than that of the surrounding parts at the same level, it tends to rise up, and, if not hindered, it continues to rise as long as its density is less than that of the medium surrounding it until it comes to the top; but, on the contrary, if its density is *greater* and continues so, it sinks to the bottom. If, therefore, the density of any part of the air, in receiving an upward motion from any cause, becomes greater, or, a downward motion it becomes less, than that of the surrounding parts at the same level, it at once tends to come back to its original position, after the disturbance ceases; but if, in rising up, the density becomes less, or sinking down it becomes greater, the tendency then is to continue in the direction started, until the conditions are changed.

The states of stable and unstable equilibrium depend upon the relation of the rate of change of temperature with change of altitude in an ascending or descending current to that in the surrounding quiet medium, since by (27), where the pressure remains the same at the same altitude in both the ascending or descending current and the quiet surrounding medium, if there is no gyratory motion, the change of density ρ depends upon the change of temperature τ in dry air, and mostly so in the case of moist air. If we let

ρ, τ=the density and temperature in ascending or descending air;
ρ', τ'=the same in the surrounding quiet air;

we shall then have in the case of the ascending saturated air the expression of (88), in which $D_h\tau$ has the value given by (83) or Table XIII; but in the case of the quiet surrounding air we shall have

$$\frac{d\rho'}{dh}=-\frac{p}{RT^2}\left(\frac{n}{R}+\frac{d\tau'}{dh}\right)$$

in which $D_h\tau'$ may have a value entirely different from that of $D_h\tau$ in the case of ascending or descending currents. Subtracting this equation from (88) we get

$$(91)\qquad \frac{d\rho}{dh}-\frac{d\rho'}{dh}=-\frac{p}{RT^2}\left(\frac{d\tau}{dh}-\frac{d\tau'}{dh}\right)$$

If $D_h\tau'>D_h\tau$, that is, if the rate of increase of temperature with increase of altitude is greater, or, in other words, if the decrease of temperature with increase of altitude is less in the surrounding quiet medium than in the ascending air, the second member is positive, and $D_h\rho>D_h\rho'$, and consequently the density of the air becomes greater in ascending and less in descending than that of the quiet surrounding air, and from what has been stated the air is in a state of stable equilibrium. If, on the other hand, the rate of decrease of temperature with increase of altitude is least in the ascending current, if started, then the density becomes less than that of the quiet air, and the air is consequently in a state of unstable equilibrium.

The values of $100D_ht$ for ascending saturated air can be obtained from Table XIII for any given pressure P and temperature τ. In order, therefore, to have the state of unstable equilibrium for such air, it is necessary that $-100D_h\tau'$, that is, the rate of decrease of temperature with increase of elevation in the undisturbed surrounding air shall be less than that in the ascending current given in Table XIII.

In the case of dry air or unsaturated air, and in all cases of descending currents, we have nearly $-100D_ht$ equal to one degree. If, therefore, in such cases, while the air is undisturbed, the temperature decreases with increase of altitude at a rate greater than 1 degree for each 100 meters, we have in all cases the unstable state.

V.—DIATHERMANCY AND TRANSPARENCY OF THE ATMOSPHERE.

Relations between heat, light, and chemical rays.

39. After entering the earth's atmosphere a part of the solar rays is reflected, another part absorbed, and the remainder is transmitted through to the earth's surface. Heat, light, and the chemical effects of the solar or other rays, are now supposed to be different effects simply of the same radiations, and not due to three distinct classes of rays. A certain amount of energy, or capacity for producing the effects called heat, light, and chemical changes, is radiated from the sun or other body, but the different effects, and their ratios to one another, depend upon the nature of the body or substance receiving them. In some bodies the whole energy is spent in producing heat, while in others a

part causes chemical changes, and consequently the heating effect is less. Light requires an eye and optic nerves, and is simply the effect of the radiated energy upon these. Although heat, light, and chemical changes are simply different effects of the same radiations, yet it is usual, on account of its convenience, to speak of heat, light, and chemical rays as if these were the effects of three distinct classes of rays, but it must be understood that they are simply different effects of the same thing. Solar and other radiations, after passing through certain media, lose more or less, by absorption and reflection, the quality or capacity of producing the effect of heat, light, or chemical changes, so that these effects of the rays, after having passed through, are less than before. *Diathermancy* is that quality of the medium which permits the heat-producing capacity of the radiations, or, in other words, the heat rays, to pass through it, and is said to be more or less diathermanous, according to the proportion of heat transmitted. *Transparency* is the same with regard to the light-producing capacity, or rays of light, which diathermancy is with regard to heat.

We have pretty definite and certain means of measuring the amount of heat received from solar and other radiations before and after having passed through a given medium, and therefore of determining the diathermancy of the medium. The measurements of light and of chemical effects are comparatively very vague and uncertain, since they are mostly mere estimates based upon the comparisons of different shadows and tints produced in a given time by chemical changes. The transparency, therefore, of a medium, and its capacity of transmitting the quality of producing chemical changes, cannot be so accurately determined as its diathermancy.

Reflections of the atmosphere and their effects.

40. If there were no atmosphere, or if this were perfectly diathermic and transparent, no heat or light would reach us from the sky except that received from the direct rays of the heavenly bodies, and the whole sky would appear entirely dark except the disks of the sun, moon, and planets, and the lights of the stars would seem to proceed from mere points without any scintillations. No terrestrial objects would be visible except those receiving the direct rays of the heavenly bodies or these rays reflected from other terrestrial bodies; all shade would be almost total darkness, and the differences between sunshine and shade temperatures would be very much increased.

The part of the rays of the heavenly bodies which, after entering the atmosphere, is lost by reflection in passing through it, is reflected and re-reflected many times in all directions, and gives rise to the diffuse heat and light of the atmosphere or sky. It is only from these irregular reflections in the atmosphere that bodies in the shade, protected from all direct radiations and reflections from other bodies, are so illuminated as to be visible, and that the sky is seen under different aspects, and does not always appear absolutely dark. If it were not for this irregular reflection we would have no twilight, but immediately after the setting of the sun everything would be involved in midnight darkness, and this would continue until the first direct rays of the sun would be received in the morning. As the twilight is still perceptible a considerable time after sunset, it is evident that these irregular reflections are extended to a long distance before the light becomes so weakened as to be imperceptible.

The following table contains the results of experiments made by Mr. Charles H. Williams on the intensities of twilight, compared with the light of sunset:

Minutes after sunset.	Per cent. of light compared with sunset.	Minutes after sunset.	Per cent. of light compared with sunset.	Minutes after sunset.	Per cent. of light compared with sunset.
0	1.000	12	.228	24	.031
1	.950	13	.200	25	.026
2	.817	14	.177	26	.021
3	.752	15	.143	27	.015
4	.655	16	.128	28	.012
5	.597	17	.101	29	.010
6	.516	18	.094	30	.009
7	.466	19	.079	31	.007
8	.407	20	.064	32	.006
9	.337	21	.055	33	.005
10	.290	22	.044	34	.004
11	.261	23	.038		

The experiments were made at Boston on 10 different clear days between the middle of November and the middle of January. From these data the average depths below the horizon corresponding to the different times after sunset can be very nearly obtained.[28]

On account also of the irregular reflections there is never very great darkness in the center of shadow of a total eclipse of the sun, although this center may be more than 200 miles from any part of the atmosphere illuminated by the direct rays of the sun. A sensible portion of light, however, may be received in this case from the corona. Another effect of these reflections is that the coldest part of the night, on the average, is a little before sunrise. For some time previous heat is received in this way, which has a sensible effect upon the temperature, just as the twilight, which enables us to see for some time before receiving the direct rays of the sun.

Diathermancy and transparency different for different wave-lengths.

41. The diathermancy and transparency of the atmosphere are different in the rays of different wave-lengths or colors, being for the most part greatest for the rays of longer wave-lengths at the red end of the spectrum, and less for those of the shorter wave-lengths toward the

other end. They both also vary very much in all kinds of rays with different states of the atmosphere, being greatest in very clear weather, but they gradually diminish with increasing haziness, and when the atmosphere is cloudy comparatively little heat or light is transmitted.

When the atmosphere is very diathermanous and transparent, as in very clear weather, most of the rays, especially from a vertical sun, pass through to the earth's surface, the remainder being either absorbed or reflected. There is then very little irregular reflection, especially of the rays of strong penetrative power of the red end of the spectrum, and the disks of the sun and moon appear bright, because a large proportion of all the rays come directly to the eye. The amount of reflected light then coming from the sky is small, and being mostly of the rays most readily reflected, belonging to the blue end of the spectrum, the color of the sky appears a very dark blue. The intensity of both the heat and light of the direct rays is then very great, but shadows are dark and cool, and we have a condition approximating to that just described in the case of no atmosphere, or of a perfectly diathermic and transparent one. But as the atmosphere gradually becomes less diathermic and transparent the sky appears first bluish, and then, after rays of all colors are reflected, of a whitish appearance; but the intensity of both the heat and light of the direct rays of the sun is diminished, and the contrast between sunshine and shade, both with regard to heat and light, is less. At times only the rays of the red end of the spectrum, having a strong penetrative power, are able to come through, and then the solar and lunar orbs appear red, and likewise thin clouds through which the rays penetrate, because red rays only reach us. This is especially the case when the sun or moon is near the horizon, and when, consequently, the rays have a longer space of atmosphere to pass through.

Effect of temperature of the source of radiation upon diathermancy and transparency.

42. According to the law of Delaroche, confirmed by the experiments of Melloni[27] and others, (§70), the higher the temperature of the source of heat rays the greater is the penetrative power of these rays through diathermanous bodies; and it is also well known that while the heat of the sun passes through a plate of glass with only small loss, and mostly by reflection, the heat rays of terrestrial dark bodies of high temperature, and even those of a fire, are almost wholly intercepted by it. Hence the heat of the sun which penetrates through the atmosphere and is absorbed by the earth's surface, is not radiated back with the same facility, being then the radiation of a dark body of low temperature, but a greater proportion of it is absorbed. For this reason the temperature of the earth is much greater than it would be if it had no atmosphere, and no doubt greater than that of the moon, which is supposed to have no atmosphere. The earth's atmosphere, therefore, serves

as a covering to protect it from the cold which would result if its heat were too freely radiated into space, just as a blanket or furs keep the heat from being conducted and radiated too freely from the bodies of animals.

Effect of aqueous vapor on diathermancy.

43. That the almost constant changes which take place in the diathermancy of the atmosphere are due in some way to the aqueous vapor contained in it is very evident, but whether they result mostly from changes in the amount of vapor in its purely gaseous state, or to partial condensations of it, which give rise to haziness and incipient cloudiness when the air is nearly saturated, is still an undecided question. On account of the strong penetrative power of the solar rays, small changes in the quantity of vapor in the air, so long as it is not sufficient to cause considerable haziness, has little effect upon the intensity of solar radiation, as it reaches the earth's surface after having passed vertically through the whole depth of the atmosphere, though about one-fourth part of the original intensity is lost in passing through in clear weather. The difference in the intensity of the solar rays at the earth's surface at sea-level when the atmosphere is very clear and when it is somewhat hazy is small, and therefore the whole diminution of intensity in passing through is due mostly to the pure atmosphere. With regard to the radiations from terrestrial bodies of comparatively low temperature, as in the case of nocturnal radiation up through the atmosphere, the effect of purely gaseous vapor upon the diathermancy of the atmosphere may be very much greater.

44. According to the experiments of Dr. Tyndall[29] on the diathermancy of a small portion of air contained in a tube, with regard to heat radiations from terrestrial sources, the adiathermancy of clear air depends almost entirely upon the aqueous invisible vapor in it, seventy times as much heat, according to the result of the experiments, being absorbed by it as by the dry air through which the rays pass. This result, however, differs very much from that which had been obtained by Magnus in experiments on the same subject, and this gave rise to considerable discussion between these physicists, Magnus maintaining that the absorption of heat in Tyndall's experiments was by a film of condensed vapor on the inside of the tube through which the rays passed. And this seems really to have been the case, according to experiments which have since been made to verify the results.

Hoorweg,[30] a few years ago, made a number of experiments the results of which are in accordance with those which had been obtained by Magnus, and which seem in a great measure to subvert those obtained by Tyndall. Still later the results of Hoorweg have been confirmed by Dr. Buff[31], so that it seems now to be conclusive that the adiathermancy of air, even to heat radiated from terrestrial sources, is not mainly due to the invisible aqueous vapor in it.

45. From the discussion by Dr. Neumayer[32] of the numerous obser-

vations made at Melbourne on nocturnal radiation, it seems that aqueous vapor has considerable effect in some way upon the diathermancy of the atmosphere to terrestrial heat radiations, but that this depends rather upon the relative humidity than upon the actual amount of vapor in the air; that is, that the diathermancy is diminished mostly when the air is nearly saturated, whether the quantity of vapor required for saturation is much or little. The observations, more than 4,000 in number, were made day and night with a radiation instrument, consisting of a spirit minimum thermometer, the bulb of which was carefully adjusted in the focus of a parabolic reflector, nicely polished and silvered, 6.4 inches wide and 2.4 inches deep. This reflector was put into a box filled with cotton and placed in a house, keeping out the rays of the sun, but in such a manner that the zenith over the instrument remained perfectly free for a space of about 38°.

The extreme values of his table of results, when the observations were classified with reference to relative humidity, are as follows: For relative humidity 22.417 per cent. and temperature 84.83° F., the difference between the air and radiating thermometer was 4.79°, while for relative humidity 96.643 and temperature 43.33° F. it was only 2.55°, thus showing that the smaller the relative humidity the greater the nocturnal radiation into space, and consequently the greater the diathermancy of the atmosphere to the heat of such radiation.

The discussion by General R. Strachey[33] of the observations made on nocturnal terrestrial radiation at Madras, in 1841–'44, shows that the greater the quantity of aqueous vapor in the atmosphere the less the fall of temperature of the air from 6.40 p. m. to 5.40 a. m., which indicates that the diathermancy of the air to such radiations is decreased with increase of the vapor tension. The extreme values in the table of results are: With a vapor tension of 0.888 inch the fall of temperature was 6° F., while with a vapor tension of 0.435 inch it was 16.5°. But this perhaps can only be regarded as a verification of Dr. Neumayer's results, since in general the greater the tension the greater also is the relative humidity, and both together simply show that aqueous vapor in some way diminishes the diathermancy of the atmosphere to terrestrial heat radiation.

The same observations were discussed by Mr. Park Harrison[34] with regard to the proportion of cloud in the sky instead of the quantity of aqueous vapor in the air. From this discussion it was found that a degree of cloudiness of .08 reduced the nocturnal fall of temperature about one-fourth of that which took place in a perfectly clear sky. But where there is cloud or haze in the upper strata of the atmosphere, there is in general greater vapor tension in the lower strata near the earth's surface, so that the effect, although undoubtedly due in great measure to the cloud, may still in some measure depend upon the corresponding increased amount of uncondensed vapor in the air beneath the clouds, and therefore be regarded as being, in some measure, only a confirmation of what has been obtained from the preceding discussions. In fact, great relative humidity, high vapor tension, and haze and cloud, go somewhat together, so that it is difficult to separate the effects and determine how much belongs to each; and all that can be inferred from the preceding discussions is that aqueous vapor in some way affects the diathermancy of the atmosphere to terrestrial radiation. But it is still left in doubt whether this is due mostly to the quantity of vapor, or to incipient condensation into haze and clouds, when the atmosphere, at least in some of its strata, is at or very near the state of saturation.

That the invisible aqueous vapor of the atmosphere is the principal absorber of the radiations of a Bunsen burner has recently been shown by a new method by W. C. Böntgen[55]. The method consists in passing the rays through air containing different proportions of aqueous vapor, by means of an apparatus devised for the purpose, and observing the increased tensions arising from the increased temperatures due to the heat absorbed. By this means he found that in pure air with dew-points of —15°, 0°, and 12°, the increase of tension from the heat rays of a Bunsen flame was respectively 0.94, 2.18, and 3.35^{mm}. The vapor tensions corresponding to these dew-points are respectively (Table X) 1.44, 4.57, and 10.43. These results indicate that with no vapor tensions there would be little, if any, heat absorption. By other methods the transmissive power of the medium is given rather than its absorbing power.

With the apparatus filled either with carbonic acid or moist air and exposed to the solar rays no sensible increase of tension was observed, even at an altitude 1,800 meters above sea level. This shows that even very impure and damp air absorbs comparatively a very small amount of the heat of solar rays passing through it. This may be due to the greater penetrative power of the solar rays, or to the fact that the kind of rays mostly absorbable by such air have been already absorbed before reaching the lower strata, but most probably to both causes.

46. According to the experiments of Mr. John Aitkin[35] haze and fog may exist in an unsaturated atmosphere if it contains much dust or products of combustion of any kind. He has in many cases produced dense fogs in unsaturated air, while in *filtered* air of the same degree of saturation, and under precisely the same conditions, no fog was seen. Of course the more nearly the air is saturated the more readily the haze and fog are formed, and most probably impure air, near, but still below, the point of saturation, is never perfectly clear on account of these condensations depending in some way upon its impurities. The dust particles may have an affinity for the vapor, or they may stand at times at a lower temperature than the air on account of the difference in radiating power, and so become nuclei of condensation, where the air is nearly but not quite saturated, for the same reason that dew is formed at night upon the surfaces of bodies which have a greater radiating power than the air, and therefore cool down to a lower temperature.

No distinction has usually been made between the absorption and reflection of heat and light upon which adiathermancy and non-transparency depend, and we have no experiments to indicate to which of these the diminution in intensity is mostly due as the rays pass through the atmosphere. It is thought that pure dry air absorbs very little of the sun's heat in its passage through to the earth. If so, the loss of intensity must be caused mostly, in this case at least, by the irregular reflections in all directions, and this view is supported by the fact that when the atmosphere is not very clear we receive a great deal of both heat and light reflected from the sky, that is, upper and surrounding atmosphere, and the part reflected out into space must be at least equal to this. And we have reason to think, from our ordinary almost daily observations, that these reflections depend in a great degree upon the hygrometric state of the atmosphere. When the atmosphere has comparatively little vapor in it, as is frequently the case in clearing weather after copious rains, we experience the effects, already referred to, § 40, resulting from an atmosphere of great diathermancy and transparency, namely, intensity of the direct heat and light of the sun, cool and dark shadows, little heat and light coming from the outside air through windows into the rooms of houses, &c. On the other hand, it is often observed, generally some time after a clearing up, and not long before another rain, that the intensity of the heat and light of the direct solar rays is not so great, although the air is clear, but that there is much more of reflected heat and light and the contrast between sunshine and shade is not so great. It seems, therefore, that the diathermancy and transparency of a clear atmosphere, at least for solar rays, must be affected mostly by the reflections, and that these depend very much in some way upon the vapor contained in it where it exists. But as this is found mostly in the lower strata near the earth's surface, and only in a small measure in the middle and upper strata of the atmosphere, its effect is small in comparison with that of the whole depth of a dry atmosphere.

Law of Bouguer.

47. The law by which the intensity of heat and light is diminished in passing through a diathermanous and transparent medium is based upon the simple and very rational hypothesis, first suggested by Bouguer, that the loss in intensity for an infinitely small space is proportional to the intensity and the mass (space into density) of the diathermic or transparent body passed through. Let

I=the intensity;
s=the space passed through;
ρ=the density;
m=the mass;
α=the rate for unit of intensity by which the intensity is lost with reference to the mass passed through.

We then have, by the principle above,

$$dI=-\alpha Idm, \text{ or } \frac{dI}{I}=d\log I=-\alpha\rho ds$$

By integration we get, in Naperian logarithms,

$$\text{(92)} \qquad log\frac{I}{A}=-\int\alpha\rho ds, \text{ or } I=Ae^{-\int\alpha\rho ds}.$$

in which e is the base of these logarithms, and in which A is the value of I where $m=0$, that is, where the rays producing heat or light first enter the diathermic or transparent body.

When the body is homogeneous with regard to its density and transmissive power, α and ρ become constants, and (92) becomes

$$\text{(93)} \qquad I=Ae^{-\alpha\rho s}$$

This expression is applicable when heat or light is transmitted horizontally through the air where α has been determined for any given density, as that of the standard barometric pressure of 760^{mm}. The value of ρ to be used then is the relative density. With either α, ρ, or s equal 0 above, we have $I=A$, and consequently no diminution of intensity.

In the case of the heat or light of the heavenly bodies passing through the atmosphere to the earth's surface, if we assume as the unit of mass passed through the whole vertical depth of an atmosphere of the standard barometric pressure of $P_0=760^{mm}$, and put

ε=the ratio of the path of oblique rays through a stratum to that of a vertical ray, which is nearly the secant of the zenith's distance z,

we shall have

$$dm=\varepsilon\frac{dP}{P_0}$$

With this value of $dm=\rho ds$ in (92) we get

$$\text{(94)} \qquad I=Ap^{\varepsilon}$$

in which

$$\text{(95)} \qquad p=e^{-\int\alpha\frac{dP}{P_0}}$$

If α is constant with regard to the different strata of atmosphere—that is, if each stratum of equal mass causes the same proportionate loss of intensity—we have

$$\text{(96)} \qquad p=e^{-\alpha\frac{P}{P_0}}$$

at any altitude where the barometric pressure is P.

When applied to solar heat radiations, A in (94) has usually been called the *solar constant* and p the *diathermancy constant*. The former, however, varies with the varying distance of the sun, being a little greater in winter than in summer; and the latter must be understood to be simply a constant in the formula with reference to the varying quantity ε, and not an absolute constant for all states of the atmosphere and situations of the stations of observation. It is seen from (95) and (96) that it varies both with P and with α—that is, with the altitude of the station and the state of the atmosphere, upon which the value of α depends very much. The value of p, therefore, may not only be very different on different days, but may change very much from one part of the day to another, as between forenoon and afternoon.

If we put $a=1-p$, a is sometimes called the ***absorption constant***, though it may depend more upon reflection than upon absorption.

If we put p_0 for the value of p when $P=P_0$ we get in (96)

$$p_0=e^{-\alpha} \tag{96'}$$

Hence from this and (96) we get, by taking the logarithms of both members of the expressions and eliminating $\alpha \log e$

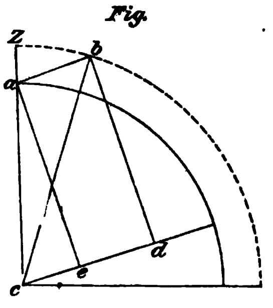
$$\log p_0=\frac{P_0}{P}\log p, \text{ or } p_0=p^{\frac{P_0}{P}} \tag{97}$$

This equation serves to reduce the diathermancy constant from the altitude where the barometric pressure is P to that where it is $P_0=760^{mm}$. But it cannot be used unless we assume, as in (96), that the different strata of the atmosphere are homogeneous with regard to transmissive power. Where this is the case p_0 may be regarded as the ***absolute*** diathermancy constant corresponding to any given state of the atmosphere, since it is independent of the altitude of the station of observation, but still dependent upon the state of the atmosphere.

48. The secant of zenith distance can be taken as the value of ε in (94), except very near the horizon. In this latter case we have more accurately

$$\varepsilon=\sqrt{1+2r+r^2\cos^2 z}-r\cos z \tag{98}$$

in which r is the earth's radius in terms of the assumed height of the atmosphere. This may be readily seen by reference to the accompanying figure, in which the line ab is the length of path through the whole atmosphere and the angle $Zab=Zcd$ is the zenith distance and ae and bd perpendicular to cd. We have



$$ab=ed=\sqrt{(bc)^2-(bd)^2}-ce$$

Considering the depth of the atmosphere unity, we have

$$ab=\varepsilon \qquad bc=1+r \qquad bd=ae=r\sin z \qquad ce=r\cos z$$

Substituting these values of the lines in the preceding equation we get (98).

As the atmosphere is not homogeneous, and has no definite known limit, it becomes a question what value of r should be used in this case. The value of $r=80$, used by Pouillet, is based upon an assumption of a height of atmosphere of about 50 miles; but this is evidently erroneous, since far below that altitude the density is sensibly nothing, and all above 15 or 20 miles can have no sensible effect upon the diathermancy of the atmosphere, taken as a whole. A better value, and perhaps the best, is $r=630$, used by Lambert, which corresponds to a height little more than that of a homogeneous atmosphere. The values of ε computed with this value by (98) are given for the corresponding zenith distances in Table XII, and also the corresponding values of the secant of z. It is seen that these values differ very little from the secant, except very near the horizon. They also are very nearly the same as those used by Bouguer and Forbes, as given by Radan[36], but for low altitudes differ considerably from those of Pouillet.

49. So far we have assumed that the rays are homogeneous—that is, that α in (92) is the same for all the rays of the spectrum. This, however, is not the case, for rays of different refrangibilities and wave-lengths have different degrees of transmissibility, as has been already stated. The rays of long wave-lengths at and beyond the red end of the spectrum are, upon the whole, less absorbed and reflected than the rays of the shorter wave-lengths toward the other end. Consequently, by (95), the values of p are different for different wave-lengths, being greatest at the extreme red end of the spectrum, where α is smallest, and least at the extreme of the other end. But the transmissibility through any medium does not depend entirely upon wave-length, since in the same group of rays of the spectrum of nearly the same wave-length there are lines in which few or none of the rays are transmitted.

The expression of (95), therefore, in which it was assumed that α in (92) is the same for all the rays, is not strictly applicable to the combined rays of the whole spectrum, though it has been generally so used. With a slight modification, however, which requires the addition of another small term, it can be made so with a very high degree of accuracy, as may be shown by a comparison of the intensities given by the formula for different zenith distances with those given by actual observation.

Effect of wave-length upon Bouguer's law.

50. If, in the case of the combined rays of the whole spectrum, we put for any group n of rays of nearly the same wave-length,

a_n=the intensity of radiation before the rays enter the atmosphere;

p_n=the corresponding diathermancy constant;

and put

$$(99) \qquad p=\frac{\Sigma a_n p_n}{\Sigma a_n}=\frac{\Sigma a_n(p+e_n)}{\Sigma a_n}$$

we shall have

$$(100) \qquad e_n=p_n-p \qquad \Sigma a_n=A \qquad \Sigma a_n e_n=0 \qquad \Sigma a_n e_n^3=0 \text{ very nearly.}$$

We shall then have by (94) and by development

$$I=\Sigma a_n(p+e_n)^\varepsilon=\Sigma a_n[p^\varepsilon+\varepsilon p^{\varepsilon-1}e_n+\tfrac{1}{2}\varepsilon(\varepsilon-1)p^{\varepsilon-2}e_n^2+\&c.]$$

This by (99) and (100) becomes very nearly

$$(101) \qquad I=A[p^\varepsilon+\tfrac{1}{2}\varepsilon(\varepsilon-1)p^{\varepsilon-2}c+\&c.]$$

in which

$$(102) \qquad c=\frac{\Sigma a_n e^2_n}{\Sigma a_n}$$

The values of $\frac{1}{2}\varepsilon(\varepsilon-1)$ for the zenith distances of the first column are contained in Table XII.

The value of p for very clear weather is about 0.77; the value of e, therefore, even in the extreme case of no transmissibility, is considerably less than unity, and in all cases it is such that the terms in the developed expression of I containing the higher powers of e must sensibly vanish, especially those containing the odd powers in which the sums of the plus and minus terms must be very nearly equal. It will be seen that even the second term in the expression of (101) is very small, except for extremely great zenith distances.

The value of A is constant for all states of the atmosphere, but that of p varies from 0 in very cloudy weather up to the value given above in the clearest weather. The value of the latter, by (92), increases with decrease in the depth of atmosphere of the same transmissive power passed through.

The values of the constants in (101), for any state of the atmosphere at a given time, can be determined from observed values of I for three different zenith distances, suitably chosen, or a great number can be taken at equal intervals during both the forenoon and afternoon, and the constants determined by the method of least squares. In the present state of actinometry there is still considerable uncertainty with regard to the true value of A, but it cannot vary much from 2.2 calories per minute, and it is probably somewhat greater.

On account of the possible small transmissibility of some of the rays through our atmosphere, we cannot assume that the value of A thus determined is strictly the true solar constant—that is, the intensity at the top of the atmosphere; for if there are any radiations from the sun which are so little transmissible as to fail to come through to the earth's surface, of course these are not taken into account, since they do not affect the observed values of I, from which the equations of condition are formed for the determination. The higher the stations, therefore, the more accurately will the value of A thus determined represent the true solar constant, but even in this case it probably falls a little short.

51. When the values of a_n and $p_n=p+e_n$ are known for a number of groups of nearly equal wave-length, comprising the whole solar spectrum, the value of the constant c can best be determined from (101).

From two series of experiments made by Professor Langley[37] at Allegheny, Pa., the approximate values of p_n have been obtained by means of averages and interpolations, as given in the following table, corresponding to the several wave-lengths in the first column. The results of the last series were based upon more observations than those of the first, and made in nearly every month of the year, so that in getting the most probable results from both double weight was given to this series.

λ	a_n	p_n	e_n	a_ne_n	$a_ne_n^2$
μ					
0.3	5	.37	−.41	−2.05	.840
0.4	10	.52	.26	2.60	.680
0.5	21	.65	.13	2.68	.357
0.6	25	.72	−.06	−1.50	.100
0.7	22	.78	.00	0.00	.000
0.8	16	.82	+.04	+ .64	.032
0.9	13	.84	.06	.78	.052
1.0	11	.85	.07	.77	.055
1.1	10	.86	.08	.80	.060
1.2	9	.87	.09	.81	.072
1.3	8	.87	.09	.72	.064
1.4	7	.88	.10	.70	.070
1.5	6	.88	.10	.60	.060
1.6	5	.89	.11	.55	.060
1.7	4	.90	.12	.48	.056
1.8	4	.90	.12	.48	.056
1.9	3	.91	.13	.39	.051
2.0	3	.91	.13	.39	.051
2.1	2	.91	.13	.26	.034
2.2	2	.91	.13	.26	.034
2.3	1	.92	.14	.14	.020
2.4	1	.92	+.14	+ .14	.020
	$\Sigma a_n=188$			$\Sigma a_ne_n=.13$	$\Sigma a_ne_n^2=2.834$

The relative intensities of solar radiation a_n, corresponding to the given wave-lengths in the preceding table, are deduced from the results of observations made by Professor Langley[37] at Lone Pine and Mountain Camp by means of interpolations from the average of both series. It is of no importance for our purpose here to know what these numbers represent in absolute intensities, and it is only important that they should be approximately correct relatively, for they are merely to be used in determining a constant in a very small term in (101).

Assuming $p=0.78$ we get from (100_1) with the values of p_n in the preceding table the corresponding values of e_n as given in the table, and from the values of a_n and e_n we get a_ne_n and $a_ne^2_n$. From the sums at the bottom of the table we then get from (102)

$$c=\frac{2.834}{188}=0.015 \tag{103}$$

We have also $\Sigma a_n e_n = 0.13$, which very nearly satisfies (100_3), as it should.

The values of p_n in the preceding table are for Allegheny. The small reduction to standard pressure or sea-level would decrease a little these values and that of p. The value of c, however, would not be sensibly changed, and for ordinary altitudes of the earth's surface it remains very nearly the same, but for high altitudes it would be considerably less, and entirely vanish at the top of the atmosphere.

It is seen from the preceding table and computations that the small wave-lengths have great weight in proportion to their intensities, and that small uncertainties in the data here make the value of c uncertain, and its value probably should be somewhat greater. But for any small change in c there is a corresponding, somewhat complementary, change in p, and the observed values of I through the whole range of zenith distances down to the horizon are nearly as well represented by (101), with a considerable range of the values of c.

A test of any law or formula is the accuracy with which it represents observations. These have generally been pretty well represented by (101) for all zenith distances with the use of the first term only in the expression, and the value of p required for this has usually been a little less on the average than that assumed above, which was shown to satisfy (99) with the values of a_n and p_n in the preceding table. Pouillet[56] obtained from hourly observations of intensities, on six clear days, a value of p ranging from 0.72 to 0.79, the average being very nearly 0.76.

If, however, we use the first term only in the expression of (101) and compute the values of I for all zenith distances with values of the constants A and p which best satisfy the observations, and compare these computed values with the observed, the residuals usually indicate that the second term in (101) has a sensible effect and is required to satisfy well the observations. If, also, the first term only in (101) is required, the values of p deduced from two equations, formed with two observed values of I taken for zenith distances suitably chosen, should give the same values. If we let I_1 and I_2 represent two observed values of intensity for smaller and larger zenith distances, respectively, the secants of which are ε_1 and ε_2, we get from (101), using only the first term,

$$(\varepsilon_2 - \varepsilon_1) \log p = \log I_2 - \log I_1 \qquad (104)$$

By computing the values of p from this expression with observed values of I_2 for different zenith distances, using the value of I_1 observed for some small zenith distance, it is found that the values of p thus obtained, in general increase with increase of zenith distance belonging to I_2. It is readily seen that this is the effect of the second term in (101) since this increases with the zenith distance and only becomes considerable for very large zenith distances. The effect of this is to increase the value of I_2 more than I_1, and hence to make p, as given by

(104), larger for greater than for smaller zenith distances belonging to I_2. This, therefore, also indicates that the second term of (101) has a sensible effect and requires to be taken into account.

When the constants A and p, therefore, are determined from equations formed from (101), using the first term only, with observed values of I taken two and two, the value of A is smaller when the second equation is formed from a value of I taken for a low altitude of the sun than it is when it is taken for a higher one, for since we have $I=Ap^{\epsilon}$ when the conditions give p larger they must give A smaller, in order to give the best values of I for all zenith distances.

It is seen, therefore, from what precedes, that Bouguer's law, which is applicable to homogeneous rays only, must be modified into the form of (101) when applied to the combination of all wave lengths of the rays, and even then there may be some rays of so little transmissibility that they are not accurately represented by that expression.

Application of the modified law of Bouguer to light.

52. Equation (101) should of course represent the observed intensities of light as well as those of heat for different zenith distances of the luminary when the transparency of the atmosphere remains the same for the several observations. In this case p becomes the *transparency* constant. Although photometry does not give the observed relative intensities of light with as much certainty in a single observation as the intensities of solar radiation are obtained, yet, on account of the great numbers of such observations which have been made, in several instances, at different times on the relative intensities of the light of the stars and combined together so as to eliminate, in a great measure, the irregularities and uncertainties in the measurements, we have data for testing the accuracy of the expression of (101) more satisfactorily than can be done in the case of solar radiation, where observations, as has been usual, of one day only are used. Ludwig Seidel[38] and Dr. Gustav Müller[39] have both formed empirical extinction tables for the purpose of reducing the brightness of an observed star at any zenith distance z to that which it would have at some other zenith distance in order to have the relative brightness or light intensity at the same zenith distance. These tables, it is known, are based upon a great number of observations taken at different times upon different stars, and treated by the graphic method of eliminating irregularities and uncertainties, and of obtaining the most probable normal values, so that the functions given in the tables may be regarded as being pretty accurate.

Putting I_0 for the intensity of light for zenith distance $z=0$, Seidel's and Müller's function φz, given in their tables, is such that we have

$$\log I = \log I_0 - \varphi z \tag{105}$$

The values are given mostly for every degree of zenith distance, but for the purpose of comparing their empirical tables with the theoretical formula of (101) those only are selected which are given in the following table in which are given for the same zenith distances the intensities as given by the formula and the two tables:

z	ϵ	$p^{\epsilon-1}$	$\phi\epsilon$	I	Seidel.			Müller.		
					ϕz	I'	$I'-I$	ϕz	I''	$I''-I$
10	1.015	.995	.000	.995	0.000	1.000	+.005	0.0004	0.909	+.004
20	1.071	.983	.001	.984	0.003	.993	.009	0.0037	.991	.007
30	1.155	.902	.002	.964	0.007	.984	.020	0.0112	.975	.011
35	1.221	.946	.004	.950	0.012	.973	.023	0.0169	.962	.012
40	1.305	.926	.005	.931	0.017	.962	.031	0.0244	.945	.014
45	1.414	.903	.006	.909	0.028	.938	.029	0.0346	.923	.014
50	1.556	.871	.010	.881	0.045	.902	.021	0.0482	.895	.014
55	1.743	.832	.013	.845	0.067	.857	.012	0.0667	.857	.012
60	2.000	.780	.019	.799	0.097	.800	+.001	0.0920	.807	.008
65	2.400	.707	.029	.736	0.140	.724	—.012	0.1276	.742	+.006
70	2.910	.622	.042	.664	0.191	.644	.020	0.1798	.661	—.003
75	3.864	.491	.067	.558	0.268	.540	.018	0.2596	.550	.008
80	5.62	.318	.104	.422	0.382	.409	.013	0.3908	.407	.015
83	7.70	.190	.124	.314	0.549	.282	.032	0.5260	.298	.016
85	10.58	.094	.125	.219	0.684	.207	—.016	0.6892	.204	.015
86	12.6	.056	.110	.166	0.741	.176	+.010	0.8164	.153	.013
87	15.5	.028	.094	.122	0.968	.108	—.014	0.9929	.102	—.020
88	19.9	.008	.049	.057	1.223	.060	+.003	1.2409	.057	....
89	26.2	.002	.020	.022	1.512	.024	+.002	....	....	....

The values of I in this table are in terms of the intensity when the luminary is in the zenith. For the intensities thus expressed, dividing the second number of (101) by Ap, the intensity at zenith distance, we get

$$(106)\qquad I=p^{\epsilon-1}+\tfrac{1}{2}\,\epsilon(\epsilon-1)\,p^{\epsilon-3}\,c$$

The two terms in this expression are computed with the assumed value of $p=0.78$, and in the last one with the value of c in (103) obtained from the results of Langley's experiments, assuming that the same value of this constant is required for both heat and light. The last term of (106) is denoted by $\varphi\varepsilon$ in the preceding table, and the sums of the numerical values of the two terms there given are the values of I given by (106), which is simply (101) adapted to the unit of intensity here adopted.

Where the unit of intensity is I_0, as here assumed, we have log $I_0=0$, and with this value (105) gives I' with Seidel's values of φz, and I'' with those of Müller. The residuals in a comparison with I obtained from (106) with the constants above are small considering the nature of the observations upon which the values of φz are based. The greatest in the case of Seidel's table is only about $\frac{1}{31}$ of the intensity of light when the luminary is in the zenith, and only half as much, or $\frac{1}{62}$, in the case of Müller's table.

The value of p, assumed above, is such as to give the best agreement between the formula of (106) and the tables, and it is remarkable that the same value is required in the case of both tables. It is the same also as that which has been shown to satisfy (99) or (100_3) in the case of heat with the values of a_n and p_n in the table in § 51, but this is no doubt in a great measure accidental.

It is seen from the column headed $\varphi\varepsilon$ in the preceding table that the last term in (106) has but little effect for small zenith distances, but becomes large beyond those of about 70°, and that for these it is very important that it should be taken into account. If we were to compare the results of Seidel's and Müller's tables with the first term merely of (106), which is an expression of Bouguer's law in the case of homogeneous rays, the value of p which would give the best comparison would of course be greater, but no value would give a satisfactory agreement. The last term, therefore, in (106), or (101), is a real term, which requires to be taken into account in the case of light as well as in the case of heat in order that the formula may represent well the tables deduced directly from observation.

The diathermancy and transparency constants, as heretofore determined, are the values of p in (101), which best satisfy observations with the last term omitted. But, as in the case of heat, so in light, if we neglect the last term in (101) we get larger values of p from (104), which is deduced from the first term merely of (101), if we use I_2 for a very large zenith distance in connection with I_1 for a small one, than we do if we use I_2 for a smaller zenith distance. With the unit used above in the case of light, if we put $\varepsilon_1=1$ we have $\log I_1=0$, and with these values (104) becomes in this case

$$(107) \qquad (\varepsilon_2-1)\log p=\log I_2$$

From this we get for a zenith distance of 60°, in which case $\varepsilon=2$, with the value of I_2 from Seidel's table, $p=.800$, and with the value from Müller's, $p=.807$. With the values of ε_2 and I_2 for 80°, we get in the two cases respectively $p=.824$ and $p=.823$; and so, still greater values for greater zenith distances.

The diathermancy and transparency constants, as usually determined, are somewhat indefinite, being a sort of average of those belonging to all the separate rays of different transmissibilities, which becomes greater or less according as greater or smaller zenith distances are used in the observations from which they are determined. And thus determined it must be greater than when determined upon the condition that it shall satisfy (99) or (100_3). But its value thus determined depends upon the value we assign to c in (101). The value of c used above in the determination of p is that of (103). If c had been regarded as an unknown constant and values of both c and p in (101) or (106) had been determined by the method of least squares, with which the formula would represent best Seidel's and Müller's tables, values a little different would, no doubt, have been obtained, and the residuals would have been a little smaller, but it is readily seen from an inspection of the residuals in the preceding table that no values of those constants, differing much from those above, could be obtained which would give smaller residuals, and which consequently would satisfy Seidel's and Müller's tables better.

53. In the trigonometrical survey of India Mr. Jacob[40], in 1837, found that in the use of the heliotropes it was necessary to enlarge the apertures *more* than in the simple ratio of the distances. By the end of the first season he had formed a scale of apertures, which, when finally corrected, stood as given in the first two columns of the following table:

Maximum distance.	Aperture.	I	ε	I'	$(I-I')$
Miles.	*Inches.*				
15	0.5	1.000	2.53	1.000	0.000
23	1.0	0.500	3.88	0.587	−0.087
33	2.0	0.250	5.56	0.248	+0.002
45	4.0	0.125	7.59	0.097	+0.028
60	8.0	0.0625	10.12	0.030	+0.032

Assuming the intensity at the distance of 15 miles as unity, the intensities I at the other distances, being inversely as the apertures required to render the heliotrope visible, would be as given in the third column.

Since ε in (101) is the distance of homogeneous atmosphere passed through by the ray in terms of the depth of a homogeneous standard atmosphere, this expression is applicable to the intensities of the light in this case by expressing the distances above in terms of the depth of an atmosphere of the pressure and temperature, on the average, which existed during the observations. Mr. Jacob states that the average of the barometer was about 27 inches, but nothing is said with regard to the temperature. We may suppose, however, that it was probably about 20° C. The height of a homogeneous standard atmosphere being 26,220 feet, (40_4), and that of any other being as the pressure and inversely as the density, or directly as the absolute temperature, we have for the height of a homogeneous atmosphere at pressure of 27 inches and temperature of 20° C. by (29)

$$l=26220\times\frac{30}{27}\cdot\frac{293}{273}=31{,}307 \text{ feet}=5.93 \text{ miles}$$

Dividing each of the distances in the first column by 5.93 we get the values of ε in the fourth column to be used in the formula of (101). Using only the first term in this expression we have

$$\log I=\log A+\varepsilon \log p \tag{108}$$

As the intensities are merely relative, such a value of log A must be used as will make $I=1$ at the first distance, for which the intensity was assumed as unity. If we assume as a first approximation log $p=-.200$, corresponding to $p=.631$, we get from the preceding expression the values I' in the preceding table, with the residuals $(I-I')$. These residuals are not large, considering the nature of the observations, but they indicate, as in the preceding example of the intensities given by Seidel's table, and in cases of solar heat intensities, that the neglected second term of (101) has a sensible effect, and that without it the intensities given by the formula are too small for low altitudes of the sun or stars or for great distances in the case of luminous terrestrial objects. A determination by the method of least squares of the constants in (101), taking into account the second term, which would best satisfy the observed intensities I, would give a still smaller value of p, and the residuals would be left very small. If this kind of observations, therefore, can be regarded as being sufficiently reliable, and the preceding result is not merely accidental, it furnishes another confirmation of the inaccuracy of Bouguer's formula, when applied to light, as it is in the case of heat, without the correction of the second term in (101).

The value of p above is smaller than that obtained by Mr. Jacob by his method, but this is due in part to his not having taken into account the decrease of the density of the atmosphere due to temperature. This value is much smaller than the value in the case in which the rays pass from the outside through all the strata of the atmosphere; but this was to be expected, and it is not at all inconsistent with the other if, as is usually supposed, the aqueous vapor of the atmosphere has any considerable effect upon its transparency and diathermancy. In the one case the rays pass through the upper cold and almost vaporless strata of the atmosphere, comprising the greater part of it, and only arrive at strata containing much vapor comparatively near the earth's surface, but in the other they pass through the whole distance near the surface, where the air, especially in the warm climate of India, contains a large amount of vapor, and where, consequently, its transmissivity for light and heat must be much diminished.

Application of Bouguer's law to the chemical effects of the rays.

54. The expression of (101) is also applicable to the intensities of the chemical effects of the solar rays; but in making this application to observations, it is found that the value of p in this case, which satisfies observation, is very small in comparison with that in the case of heat or light. Roscoe and Thorp[41] give the results of the determinations of the chemical intensity of total daylight made in the autumn of 1867, on the flat table land on the southern side of the Tagus, about 8½ miles to the southeast of Lisbon, under a cloudless sky, with the object of ascertaining the relation existing between the solar altitude and the

chemical intensity. The first part of the following table contains their results corresponding to the given zenith distances Z.

No. of observation.	Z		Chemical intensity or I.			ϵ	I'	$I-I'$	I''	$I-I''$
			Sun.	Sky.	Total.					
	°	′								
11	25	46	.221	.138	.359	1.107	.215	+.006	.362	−.003
24	28	52	.195	.132	.327	1.143	.198	−.003	.339	−.012
19	36	51	.136	.126	.262	1.250	.154	−.018	.279	−.017
22	47	47	.100	.115	.215	1.489	.089	+.011	.182	+.087
22	58	46	.052	.100	.152	1.928	.032	+.020	.080	.072
18	70	19	.023	.063	.085	2.965	.003	+.020	.012	.073
15	80	9	.000	.038	.038	5.85	.000	.000	.000	.038

Putting log $A=0.440$ and $p=.10$ in (108) it gives, with the given values of ϵ, the intensities I' in the table for the sun; and the column $(I-I')$ shows the difference between computation and observation. The plus signs of the residuals for large zenith distances, as in the case of heat, and also of light, indicate that the second term of (101) has a sensible effect, and that the observations would be much better satisfied if this term were taken into account, but this would give a still less value of p. The small value of p required to satisfy approximately the observations, indicates that at least nine-tenths of the quality in the direct solar rays which produces the chemical effects, is lost in passing vertically through the depth of the whole atmosphere. Consequently with a zenith distance of about 70°, where the rays have to pass through a depth of nearly three atmospheres, the intensity is reduced to $\frac{1}{12}$ of the vertical intensity $Ap=2.75\times.10=.275$. At the zenith distance of 80° there is no sensible effect given either by observation or the formula.

It has been shown elsewhere[52], and is readily seen from (95), that the same formula is applicable when the reflections of the atmosphere are taken into account, and that the value of the constant A is the same in both, but that the value of p is greater in the latter case. Since α is the rate for unit of intensity by which the intensity of radiation is lost, when the part of the diffuse reflections which reaches the earth is taken into account of course the rate of loss is less, and this must always be in proportion to the intensity I at the point of reflection, just as the part lost by absorption is, and consequently the whole effect is to decrease α and increase p in (95).

If now, in the case of the total intensity of the sun and sky, we use log $A=0.440$ as before, but put $p=.16$ in (108), we get for the intensities of chemical effect the values I'' in the preceding table, and for the residuals the numbers in the column headed $(I-I'')$. But here, as in the preceding case, and in all cases of heat and light, the plus signs for large zenith distances indicate that the second term of (101) has a sensible effect, and considerably greater than in the preceding case. Taking this into account the residuals would no doubt be quite small, considering the nature and the uncertainty of the observations.

Examples.

(*a*) If the solar constant is 2.2, and the value of $p=0.75$, what is the intensity I, by 94, for a zenith distance of 60°?

In this example we have $\log p=-.1249$ and $\log A=0.3424$, and $\varepsilon=2$. Hence we get $\log I=0.3424-2\times .1249=0.0926$. Whence $I=1.238$.

(*b*) With the value of c in (103) what is the effect of the second term in (101)?

In this example we have from Table XII $\frac{1}{2}\varepsilon(\varepsilon-1)=1$, and $p^{\varepsilon-2}=1$. Hence the value of the term is 0.015.

(*c*) With the solar constant 2.2, and the value of $p=0.7$ and $c=.015$, what is the value of I for a zenith distance of 70°, taking account of the first two terms in (101)?

(*d*) With an observed value of $I=0.725$ for a zenith distance of 66°, what is the value of p for homogeneous rays?

(*e*) If on the top of Pike's Peak, bar. 17.1 inches, the value of p is 0.88, what would it be at sea-level, bar. 30 inches?

VI.—FRICTION OF THE ATMOSPHERE.

Friction formulæ.

55. If strata of air or of any gas move over one another with relative velocities they suffer resistances from one another called friction, which tend to destroy these velocities and to reduce all to the same absolute velocity. This resistance is explained by the kinetic theory of gases. Since by this theory the molecules of any gas are constantly in motion, being reflected and re-reflected in all directions, and in virtue of their mass and velocity have a certain momentum, the continuous interchange of molecules between the different strata and their numerous contacts tend to equalize the velocities between the strata. It is from this that the effect of friction arises. Putting

u=the relative velocity;
h=a normal to the strata;
f=the resistance for a unit of surface,

we have

$$f=\eta\frac{du}{dh} \tag{109}$$

in which η is called the *coefficient of friction*. This is a function of the temperature which may be expressed in the form

$$\eta=\eta_0(1+k\tau) \tag{110}$$

in which η_0 is the value of η at $\tau=0$ and k is the constant rate of increase with increase of τ.

Determinations of the constants.

56. The values of the constants can only be determined by experiment. These experiments have been of two kinds; the one by means of oscillating pendulums of flat glass and metal disks, in which the rate of retardation is observed, and the other by means of the flow of gases through tubes, in which the rate of flow by volume is observed. By the former method Maxwell[46] obtained $\eta_0=.000188$, and E. Meyer[47] the same; and by the latter method Meyers' value[48] was $\eta_0=.000174$. Other experiments have given nearly the same values of η_0.

According to the theoretical deductions of Maxwell[49] η is independent of density or pressure at constant temperature, and is proportional to the "velocity of mean square" of the molecules, and hence, to the square root of the absolute temperature. The first seems to be confirmed by the experiments of Kundt,[15] but the latter has not been by any experiments. For if it were true, from the development of

$$\sqrt{T}=\sqrt{273(1+\alpha\tau}=\sqrt{273}(1+\tfrac{1}{2}\alpha\tau+\&c.)$$

we should have in (110)

$$k=\tfrac{1}{2}\alpha=\tfrac{1}{2}(.003665)=0.00183$$

But the values obtained by Meyer, Puluj, Obermayer, and Warburg range from 0.00242 to 0.0030, the average being 0.0027.[15]

The average of numerous experiments by Mr. Silas W. Holman[42] makes the value of η increase as the 0.77 power of the absolute temperature. This gives in (110)

$$k=0.77\times .003665=0.0028$$

VII.—Limit of the Atmosphere.

57. According to the laws of Boyle and Charles the pressure and density of the atmosphere, as seen from (57) and (27), cannot vanish, except for an infinite value of v, and, consequently, if these laws hold for all extremes there must be a universal atmosphere occupying all space, if there is an absolute temperature of space T. In case of no such temperature we should have p, which must here be regarded as tension, equal 0, and hence we would not have the gaseous state, to which alone the laws of Boyle and Charles apply. From a very rare and universal atmosphere the sun, moon, and planets would attract a portion nearer to themselves and form surrounding denser atmospheres.

If an atmosphere like ours pervaded all space it can be shown that the sun, moon, and planets would each attract so large a portion and form an atmosphere around themselves so dense as to refract the rays of light passing from the stars near them through their atmospheres so much as to cause great displacements in their apparent positions, which, at least in the case of the moon, could be readily observed. Besides, the sun would draw so large a portion that the density of its atmosphere would be sufficiently great at the distances of the planets to sensibly affect the periods of their revolutions in orbit. But no such displacement or change of periods is observed, and hence there cannot be a universal atmosphere of the nature of ours.

Upon the hypothesis of the universality of our atmosphere, Dr. Thiesen, of Berlin, has shown that the density of the sun's atmosphere, upon any reasonable assumption of temperature, even of 30,000° C. at the sun's surface, would be, at the distance from the sun of 0.8235 of the earth's mean distance, and so, far beyond the orbit of Venus, equal to the density of our atmosphere at the earth's surface. This is a much greater density than that which had usually been obtained upon the hypothesis of Dr. Wollaston, that equal densities of this universal atmosphere in the vicinities of the sun and planets should correspond to equal forces of gravity, the error of which hypothesis has been pointed out by Thiesen. According to this hypothesis the density of our atmosphere at the earth's surface would be found in the sun's atmosphere at the distance of only 4½ of the sun's radius.

According to the table of § 27 the atmosphere sensibly vanishes at the altitude of 60 kilometers. But this is upon the hypothesis that the laws of Boyle and of Charles hold for very low pressures and temperatures, and also that the assumed temperatures are correct for high altitudes. We have seen (§§ 21, 23) that there are considerable deviations from observation in both laws within the range of experiments, and for the very small tensions and temperatures at so great altitudes we do not know how great these deviations may be. Neither do we know how much the actual temperatures there may vary from those assumed. The atmosphere at very great altitudes, therefore, may not be nearly so rare as it would be by the table of § 27.

From twilight observations it would seem that a sensible portion of light is reflected from the illuminated part of the atmosphere at the height of 45 miles. This would certainly be impossible if the atmosphere at that altitude is as rare as indicated by the table referred to.

CHAPTER II.

TEMPERATURE OF THE ATMOSPHERE AND EARTH'S SURFACE.

I.—The Relative Distribution of Solar Radiation.

Introduction.

58. The source of all heat upon which the general temperature of the atmosphere and the earth's surface depends is the sun, and the distribution and the variations of temperature depend primarily and mostly upon those of the intensity of solar radiations. The first step, therefore, in an attempt to ascertain and to explain the distributions of temperature in the atmosphere and over the earth's surface, and the variations of temperature at different times, is to ascertain the intensity of solar radiation over the different parts of the earth's surface at different seasons of the year and times of the day.

Expressions of mean diurnal intensity of solar radiation.

59. Since the value of the solar constant, upon which the intensity of solar radiation depends, is still very uncertain, and requires further experiment and research to determine it even approximately, the absolute values of the intensity of solar radiation at any given time and place on the earth's surface cannot be given, and the best that can be done is to give the relative intensities in terms of the solar constant for the sun's mean distance. But we have seen that the heat of the sun's rays after entering the earth's atmosphere are absorbed, and dispersed by reflections in all directions, so that even in clear weather, with a vertical sun, only about three-fourths are transmitted to the earth's surface. The relative intensities, therefore, can be given for the top of the atmosphere only, or with reference to the whole amount of heat received by both the atmosphere and earth's surface. In doing this, the following notations will be adopted:

J = the vertical intensity of solar radiation;
δ = the sun's declination;
z = its zenith distance;
h = the hour angle from apparent noon;
H = the value of h at sunset;
λ = the geographical latitude of the place.

Since the normal intensity—that is, the intensity on a plane perpendicular to the sun's rays—at the top of the atmosphere is the solar constant A, we shall have for the vertical intensity, which is the intensity upon a horizontal surface,

$$(1) \qquad J=A\cos z$$

This is a maximum for a vertical sun, but it vanishes when the sun is on the horizon.

We have from a well-known trigonometrical relation, found in any treatise on spherical trigonometry,

$$(2) \qquad \cos z=\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos h$$

We therefore have

$$(3) \qquad J=A\,(\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos h)$$

From this expression the vertical intensity can be computed, with the given latitude λ and solar declination δ, for any time of day for which the hour angle is h.

It is seen from mere inspection that J contains both a diurnal and an annual inequality, the former depending upon the variable h, and the latter upon δ. The former is greatest at the equator, where $\cos\lambda=1$; and when the sun is on the equator, where $\cos\delta=1$; and it entirely vanishes at the poles, where $\cos\lambda=0$. The latter is greatest at the poles, where $\sin\lambda=1$, and vanishes at the equator where $\sin\lambda=0$. There is also a semi-annual inequality depending upon $\cos\delta$ in the last term of the expression, which has a maximum at the equator and gradually decreases toward the poles.

The expression of J with regard to the diurnal inequality is a discontinuous function which vanishes at sunset, and remains 0 until sunrise. In order, therefore, to obtain the whole amount of heat received in one day equation (3) must be integrated from sunrise to sunset—that is, from $h=-H$ to $h=H$. For this purpose, since J is the rate of receiving the solar heat, Q, we can put

$$J=\frac{dQ}{dt}=\frac{dQ}{dh}$$

if h is expressed in time.

With this expression of J in (3) it becomes

$$(4) \qquad dQ=A\,(\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos h)\,dh$$

Regarding A and $\sin\delta$ as constant during one day, as we can without sensible error, we get by integration from $h=-H$ to $h=H$

$$(5) \qquad Q=2A\,(\sin\lambda\sin\delta\cdot H+\cos\lambda\cos\delta\sin H)$$

in which the hour-angle H is expressed in arc in terms of the radius.

To obtain the mean vertical intensity of solar radiation for one day we must divide the whole heat received in a day by the number of units of time in a day. But as H above is expressed in terms of the radius we must divide by the time or hour-angle for a whole day, expressed also in terms of the radius, which is 2π. Dividing, therefore, both members of the equation above by 2π we get

$$(6) \qquad J=\frac{Q}{2\pi}=\frac{1}{\pi}A\,(\sin\lambda\sin\delta\cdot H+\cos\lambda\cos\delta\sin H)$$

The hour angle H for sunset in this expression is a function of δ, and could be computed for any day of the year. But this would be inconvenient, and it is therefore desirable to have J in a function of δ only by substituting for H its value in a function of δ.

60. From right-angled spherical trigonometry we have

$$(7) \qquad \cos H=-\tan\lambda\tan\delta=-\frac{\tan\lambda\sin\delta}{\sqrt{1-\sin^2\delta}}$$

Dividing the last term in the expression of J above by $\cos H$, and multiplying it by its equal

$$-\tan\lambda\tan\delta=-\frac{\sin\lambda\sin\delta}{\cos\lambda\cos\delta}$$

we get

$$(8) \qquad J=\frac{1}{\pi}A\sin\lambda\sin\delta\,(H-\tan H)$$

If we now put

$$H=\frac{\pi}{2}-\left(\frac{\pi}{2}-H\right)$$

and express the last term, $(\frac{1}{2}\pi-H)$, in a function of its sine, which is $\cos H$, since the sine of an angle is the cosine of its complement, we get

$$H=\frac{\pi}{2}-\left(\cos H+\frac{1}{2}\,\frac{\cos^3 H}{3}+\frac{1\,.\,3}{2\,.\,4}\,\frac{\cos^5 H}{5}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,\frac{\cos^7 H}{7}+\&c.\right)$$

By putting

$$\tan H=\frac{\sin H}{\cos H}=\frac{\sqrt{1-\cos^2 H}}{\cos H}$$

and developing the numerator of the last form, we get

$$\tan H=\frac{1}{\cos H}\left(1-\frac{\cos^2 H}{2}-\frac{1}{1\,.\,2}\,\frac{\cos^4 H}{2^2}-\frac{1\,.\,3}{1\,.\,2\,.\,3}\,\frac{\cos^6 H}{2^3}\right.$$
$$\left.-\frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3\,.\,4}\,\frac{\cos^8 H}{2^4}-\&c.\right)$$

We therefore have from this, and the last expression of H above,

$$(9) \qquad H-\tan H=\frac{\pi}{2}-\frac{1}{\cos H}-\frac{1}{2}\cos H-\frac{1}{24}\cos^3 H$$
$$-\frac{1}{80}\cos^5 H-\frac{5}{896}\cos^7 H \quad \&c.$$

From the development of the denominator of the last form of the second member of (7), we get

$$\cos H = -\frac{\sin\lambda \sin\delta}{\cos\lambda}\left(1+\frac{\sin^2\delta}{2}+\frac{1\,.\,3}{1\,.\,2}\,\frac{\sin^4\delta}{2^2}+\frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\,\frac{\sin^6\delta}{2^3}-\&c.\right)$$

The reciprocal of this is

$$\frac{1}{\cos H} = -\frac{\cos\lambda}{\sin\lambda \sin\delta}\left(1-\frac{\sin^2\delta}{2}-\frac{1}{1\,.\,2}\,\frac{\sin^4\delta}{2^2}-\frac{1\,.\,3}{1\,.\,2\,.\,3}\,\frac{\sin^6}{2^3}-\&c.\right)$$

With these expressions of cos H and of its reciprocal we get from (9) an expression of $(H-\tan H)$, which, being substituted in (8), gives:[51]

$$(10)\qquad J=\frac{1}{\pi}A\left|\begin{aligned}&\frac{1}{2}\pi \sin\lambda \sin\delta\\ &+\cos\lambda\left(1-\frac{\sin^2\delta}{2}-\frac{1}{1\,.\,2}\,\frac{\sin^4\delta}{2^2}-\frac{1\,.\,3}{1\,.\,2\,.\,3}\,\frac{\sin^6\delta}{2^3}-\&c.\right)\\ &+\frac{\sin^2\lambda \sin^2\delta}{2\cos\lambda}\left(1+\frac{\sin^2\delta}{2}+\frac{1\,.\,3}{1\,.\,2}\,\frac{\sin^4\delta}{2^2}+\&c.\right)\\ &+\frac{\sin^4\lambda \sin^4\delta}{24\cos^3\lambda}\left(1+\frac{3}{2}\sin^2\delta+\&c.\right)\\ &+\frac{\sin^6\lambda \sin^6\delta}{80\cos^5\lambda}\left(1+\&c.\right)\end{aligned}\right.$$

From this expression with the value of δ taken from a solar ephemeris, the value of J can be computed for any latitude λ and day of the year in terms of the normal intensity A. But this varies inversely as the square of the sun's distance, and is consequently, as is δ, a function of the sun's longitude, and so a function of the time.

Harmonic expressions of the intensity.

61. Since both A and δ in (10) are functions of the time the expression of J may be obtained in a harmonic function of the time t. Let us put

l=the sun's longitude;
ω=the obliquity of the ecliptic;
a=the mean distance of the sun;
r=the radius vector of the earth in its orbit;
e=the eccentricity of the earth's orbit;
$\tilde{\omega}$=the longitude of perihelion;
l'=the sun's mean longitude when $t=0$.
i=the mean velocity of the sun in longitude.

We then have from well-known trigonometrical and astronomical relations:

$$(11)\qquad \begin{aligned}&\sin\delta=\sin\omega \sin l\\ &l=it+l'+2e\sin(it+l'-\tilde{\omega})\\ &\sin l=\sin(it+l')+e\sin(2it+2l'-\tilde{\omega})-e\sin\tilde{\omega}\\ &\frac{a^2}{r^2}=1+\tfrac{1}{2}e^2+2e\cos(it+l'-\tilde{\omega})\end{aligned}$$

From the last of these we get

(12) $$A = A_0 \frac{a^2}{r^2} = A_0[1+\tfrac{1}{2}e^2+2e\cos(it+l'-\tilde{\omega})]$$

in which

A_0 = the value of A at the sun's mean distance, a'

According to Leverrier's *Tables du Soleil*, we have for Washington mean noon, January 0, 1882,

(13) $$l'=280^\circ\,16' \qquad \tilde{\omega}=280^\circ\,54'$$
$$i=0.9856^\circ=.01720 \text{ in arc} \qquad e=0.0168$$

62. By means of the preceding relations and values the expression of J in (10) may be put into the following form:

(14) $$J=A_0(C_0+C_1\cos(it-c_1)+C_2\cos(2\,it-c_2)+\&c.$$
$$=A_0\Sigma C_s\cos(sit-c_s)$$

in which $c_0=0$ and the other constants C_s and C_s have the numerical values in the following table for each tenth degree of latitude:

Latitudes.	C_s and c_s.							Southern hemisphere.			
	C_0.	C_1.	C_2.	C_4.	c_1.	c_2.	c_4.	C_1.	C_2.	c_1.	c_2.
					° ′	° ′	° ′			° ′	° ′
0°...	.3053	.0101	.0131	.0001	0 30	159 8	318 24	.0101	.0131	0 30	159 8
10°...	.3010	.0247	.0136	.0001	166 22	160 4	318 24	.0444	.0113	352 2	158 0
20°...	.2885	.0585	.0128	.0001	167 46	161 4	318 24	.0776	.0084	350 56	156 4
30°...	.2682	.0909	.0107	.0000	168 30	162 31	.. ...	.1084	.0042	350 38	150 24
40°...	.2411	.1200	.0069	.0001	168 50	165 54	138 24	.1359	.0016	350 23	6 44
50°...	.2085	.1456	.0009	.0002	169 4	207 36	138 24	.1592	.0092	350 12	343 6
60°...	.1732	.1667	.0089	.0004	169 11	332 0	138 24	.1780	.0201	350 6	342 14
70°...	.1413	.1824	.0270	.0017	169 17	335 34	128 34	.1917	.0393	350 0	340 50
80°...	.1509	.1909	.0909	.0017	169 16	338 37	138 24	.2009	.1037	350 1	339 50

For the expressions of these constants, from which the numerical values here given are computed, and the method by which they are obtained, the reader is referred to Professional Paper of the Signal Service, No. XIII.[52]

The expression of (14), together with the numerical values of the constants in the table, is important in the comparisons with temperature expressions of the same form on different latitudes of the globe, in order to observe the relations between the changes of the constants in (14) for the different latitudes and those of the corresponding constants in the temperature expressions.

The expression of (14) is not applicable within the polar circles to the times in which the sun does not rise and set, and consequently within these circles it is not a continuous function through the year.

The mean solar intensity for the year for any latitude up to the polar circle is expressed by the first term of (14) $A_0\ C_0$; and hence, according to the values of C_0 in the table, it decreases with increase of latitude. Since (14) has been deduced from (10), and the latter from an integration extending from sunrise to sunset, (14) is not applicable within the polar circles while the sun does not rise and set, and consequently $A_0\ C_0$ does not express the mean annual intensity there.

It is seen from the numerical values of C_1 that the annual inequality of solar intensity is very small at the equator but rapidly increases with increase of latitude. The values of C_2 indicate a small semi-annual inequality at the equator which gradually decreases with increase of latitude up to the parallel of about 50° when it becomes very small, and then increases again toward the poles. The inequality having a period of one-fourth of a year, it is seen from the values of C_4, is so small that it can generally be neglected.

The values of C_1 for the southern hemisphere, compared with those of the northern, show that the annual inequality in the intensity of solar radiation is much greater in the former than in the latter. This arises from the circumstance that the sun is nearest the earth in the summer of the southern hemisphere.

Times of maxima of the inequalities.

63. If we put

$T_s =$ the times of maxima of the several components,

these, by (14), are determined by the conditions that $s\,it = c_s$, which give

$$T_s = \frac{c_s}{si}$$

This gives for the time of maximum of the annual inequality on the parallel of 45°, where by the preceding table $c_1 = 168°\ 57'$.

$$T_1 = \frac{c}{i} = \frac{168°\ 57'}{0.9856} = 171.4 \text{ day.}$$

Since the era, or time $t = 0$, in the table is Jan. 0, noon, Washington time, this indicates that the maximum occurs in the afternoon of the 171st day of the year, Washington time, or by the Table XI, during the afternoon of the 20th of June.

For lower latitudes, since there the values of c_1 in the preceding table are smaller, the time of the maximum is a little earlier.

Since the other components sensibly vanish in the middle latitudes, as is seen from the values of C_2 and C_4 in the preceding table, T_1 may be regarded as the resultant maximum of all the components.

Expressions of intensity where the sun does not rise and set.

64. The expression of J for the time the sun does not rise and set is obtained by integrating (4) for 24 hours, that is, from $h = -\frac{1}{2}\pi$ to $h = \frac{1}{2}\pi$, where h is expressed in arc in terms of the radius, instead of

from sunrise to sunset, and dividing by 2π, just as in the preceding case. We get, of course, the expression of (6) except that H in this case becomes π, and consequently, $\sin H=0$. We have, therefore, in this case,

$$J=A \sin \lambda \sin \delta$$

This, by means of the relations of (11) and (12), can be put into the form

$$(15) \qquad J=A_0 \Sigma C_s \sin (s\, it-c_s)$$

in which

$$(16) \qquad C_1=\sin \lambda \sin \tilde{\omega} \qquad C_2=2 e \sin \tilde{\omega} \qquad c_1=-l'=79^\circ\, 36'$$
$$c_2=\tilde{\omega}-2 l'=80^\circ\, 6'$$

This expression of J, it must be borne in mind, can be used for the time only that the sun does not set, and consequently within certain limiting values of t. These limiting values are the times when the polar distance of the place is equal to the sun's declination—that is, when $\delta=\frac{1}{2} \pi-\lambda$. Hence we get from ($11_1$) and ($11_2$)

$$\pm \sin (\tfrac{1}{2}\pi-\lambda)=\sin \omega\, [\sin (it+l')+e \sin^2 it+2 l'-\tilde{\omega}-e \sin \tilde{\omega}]$$

as a condition for determining these limits for any latitude λ. Substituting the numerical values of $\sin \omega$, $\sin \tilde{\omega}$, l', and $\tilde{\omega}$, it becomes

$$(17) \qquad \pm \frac{\sin (\frac{1}{2} \pi-\lambda)}{0.3981}-.0164=\sin (it+280^\circ\, 24')+.0168 \sin (2\, it+79^\circ\, 54')$$

The negative of the double sign must be used for the southern hemisphere. For each hemisphere there are two values of t which satisfy (17), which are the limiting values required. These are likewise the limiting values of t in (14), where it is applied within the polar circles, for the time that the sun rises and sets.

Table of vertical and normal intensities.

65. In Table XI are contained the vertical intensities of solar radiation J, in terms of the mean solar constant A_0, for each tenth parallel of latitude of the northern hemisphere, and for the first and sixteenth day of each month, computed by (14), with the values of the constants C_s and c_s in the table of § 62, for all latitudes below the polar circle and for latitudes within the polar circle during the time that the sun rises and sets. Within the polar circle for the time that the sun does not set, determined by the condition of (17), they have been computed by (15), with the values of the constants given by (16). The table also contains the values of the angle it for the given dates, and likewise the values of A in terms of A_0, both of which will often be found very useful and convenient. All values for intermediate dates of the table can be

readily obtained by interpolation. The numbers between the horizontal lines in the columns for latitude within the polar circle belong to the time during which the sun does not set, the blank places to the time it does not rise.

In the southern hemisphere the values of J in Table XI differ for any given time of the year on account of the differing values of the constants C_1, C_2, c_1, and c_2 in the table of § 62. For the mean of the year, however, the terms in (14), containing these constants, are eliminated, so that the mean intensities for the year, contained at the bottom of Table XI, are the same for both hemispheres.

The era, or time $t=0$, in the table and the formulæ, is noon of January 0, for the average of four or a long series of years, but for single years, where great accuracy is required, the following may be used:

January 1^d 3^h a. m., for leap year.
January 0 9 a. m., for the first after leap year.
January 0 3 p. m., for the second after leap year.
January 0 9 p. m., for the third after leap year.

In the column of dates one day must be added in January and February in leap years.

For the *absolute* vertical intensities J, and normal intensities A, the numbers in the table, which express simply the relative intensities, must be multiplied into A_0. There is, however, considerable uncertainty yet with regard to its true value, and for the present we must be content with approximate values, each one using that which he considers the most probable value.

Equal amounts of heat received by the whole earth in equal times.

66. From Kepler's law of equal areas in equal times we have, using the notation of § 61,

$$\frac{1}{2}r^2dl=cidt \tag{18}$$

in which c is a constant depending upon the eccentricity of the earth's orbit. The integration of this for the time of one revolution T, since iT in arc is equal to 2π, gives

$$\int\frac{1}{2}r^2dl=\pi a^2\sqrt{1-e^2}=2\pi c$$

the integral of the first member being the area of the ellipse of the earth's orbit, which is $\pi a^2\sqrt{1-e^2}$. From this we get $c=\frac{1}{2}a^2\sqrt{1-e^2}$, and with this value of c (18) gives

$$\frac{a^2}{r^2}=\frac{1}{i\sqrt{1-e^2}}\frac{dl}{dt}$$

With this value of $a^2{:}r^2$ in the first form of the expression of A in (12) we get

$$A=A_0\frac{1}{i\sqrt{1-e^2}}\frac{dl}{dt} \tag{19}$$

Hence the normal intensity of solar radiation at any time is proportional to the rate of change of longitude, equal amounts of heat being received upon each unit of normal surface for equal parts of longitude passed through in any part of the orbit. On account of the spherical figure, or nearly so, of the earth, the amount of normal surface for the whole earth, receiving the sun's heat, which is the area of a great circle of the earth, is the same at all times. The amount of heat, therefore, received by the whole earth is proportional to the angular change of the earth in orbit, or of the sun in longitude.

This, however, is not strictly true where there is a secular change in the eccentricity e of the earth's orbit, and this change, in the course of a very long series of years, might cause a sensible change in the annual mean of A, and consequently in the amount of heat received by the earth during a year. The greater the eccentricity the greater the amount of heat received, but it is seen from (19) that the change in e would have to be considerable to produce much effect.

Since the amount of heat received by the earth is proportional to the longitude passed through, and not to the time, and the sun while north of the equator has the same relation to the northern hemisphere that it does to the southern hemisphere while it is south of the equator, it follows that the same amount of heat is received in the course of a tropical year by both hemispheres.

Expressions of intensity within the earth's atmosphere.

67. For any body within the atmosphere or at the earth's surface, instead of (3), we have approximately for the vertical intensity at any hour h, neglecting the small terms in (101) Chapter I,

$$J=Ap^{\varepsilon}\,(\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos h) \tag{20}$$

As both ε and h are functions of t, the diurnal inequalities of intensity, although (20) is a discontinuous function, we know from actual observation, can be represented by

$$J=\Sigma K_s\cos\,(s\,nt-k_s) \tag{21}$$

in which

$n =$ the rate of change of the hour angle h,

and in which K_s and k_s are constants which can only be determined by observation. Although J entirely vanishes from sunset to sunrise, yet it can still be represented very nearly by (21) without taking very many terms, for as there are two constants K and k for each inequality, the bi-hourly observations for 24 hours could be accurately represented by six terms of such an expression besides the constant K_0 independent of t, and the deviations between these intervals would be small. The expression is rounded off by the effect of p^{ε}, which causes the normal intensity to become small as the sun approaches the horizon, so that (21) is more convergent at the earth's surface than it would be at the top of the atmosphere, or at the earth's surface in case of no atmosphere.

From an inspection of (20), of which this is a development with regard to the variable $h=nt$, it is evident that K_s must in general, especially K_1, be greatest at the equator where the extreme values of p^ε differ the most and cos Λ is greatest, and gradually decrease to the poles, where they all vanish. And since the difference between the extreme values of p^ε are greater in summer than in winter, the values of K_s must be greater in the former season than in the latter. They must also decrease somewhat as p^ε decreases and vanish when p vanishes, and hence vanish in very cloudy weather, in which we have $p=0$.

In the case of the mean diurnal intensity, which would be obtained from the integration for one day of (20) put into the form of (4), the inequalities depending upon nt, as expressed in (21), vanish, and we have left only the mean diurnal intensity and the inequalities depending upon it, of which both ε and δ in (20) are functions. The expression of J, therefore, can be put into the form

$$J=\Sigma K_s \cos\,(s\ it-k_s) \tag{22}$$

similar to that of (14), with inequalities of the same periods, but in which the values of the constants are different. As we approximate to the top of the atmosphere, however, where the value of p^ε becomes unity, the values of K_s and k_s approximate to those of A_0C_s and c_s in (14), and at the top become the same.

Since this is the development of (20) with regard to the variable l, which by (11_2) is a function of it, K_1, in this expression, must be small at the equator where $\sin\lambda=0$, and must gradually increase with increase of latitude. The values of K_s must also be in general less than those in (21), because in middle latitudes where $\sin\lambda$ and $\cos\lambda$ are nearly equal, the coefficient of $\sin l$ in (11), which is the expression of $\sin\delta$, is much smaller than the average of $\cos\delta$ in the last term of (20) upon which (21) depends. In high latitudes, however, the reverse may be true.

II.—THE CONDITIONS DETERMINING TEMPERATURE.

Introduction.

68. With the relative intensities of solar radiation for different times and places on the earth's surface, we are far from knowing the absolute or even the relative temperatures at different times and places resulting from these intensities, for there is no proportionality, or other known and simple relation, between them.

We may expose a black-bulb and bright-bulb thermometer, a piece of glass or other diathermanous body, and a number of opaque bodies of different qualities and shapes, in a vacuum to the same intensity of the solar rays; and yet the static temperatures of all these bodies will differ, the temperature of the black bulb will be greater than that of the bright, and the temperature of the latter greater than that of the piece of glass, and the temperatures of all the other bodies will differ according to their different qualities and shapes. If they are all exposed in a similar manner in the open air there will still be differences, but not so great, the conduction and convection of the air having an equalizing effect, especially when in motion.

Again, if these same bodies be exposed to a varying intensity, such as is represented by (21) or (22), the corresponding inequalities of temperature will have relations to one another very different from those of the inequalities of intensity of solar radiation, the ranges of the inequalities and times of maxima and minima being very different in the different bodies. The ranges of inequality of larger bodies will in general be greater than those of smaller ones, and those bodies of the same shape and size but of different qualities, and of bodies of the same size and quality but of different shapes, will differ very much one from another, all exposed to the same inequalities of solar radiation. Much also depends upon the surroundings of the bodies.

The important problem to be solved here is, having the conditions of shape, size, quality, &c., of a body, its surroundings, and the intensity of solar radiation to which it is exposed and its variations, as expressed in (21) or (22), to determine the resulting temperature and its variations. The problem is mostly too complex to obtain absolute quantitative results, but very important general relations can be established between the intensity of solar radiation and the resulting temperature depending upon the conditions which enter into these relations. The principal circumstances upon which they depend are the laws and rates of radiation and absorption of the bodies, their size, shape, and capacity for heat, the temperature of surrounding bodies, and when in contact with other solid bodies or immersed in fluids, upon the conducting power of the solids and the convective power of the fluids for heat with which they are in contact. Where also the intensity of solar radiation is variable, subject to inequalities of longer or shorter duration, these relations depend very much upon the periods of the oscillations of intensity.

Laws of radiation and absorption.

69. Every part of the surface of a body radiates heat in all directions from it. The intensity of the radiation is a function of the quality of the radiating surface, called its radiating power, upon its temperature, and also upon the angle of emission with reference to the normal to the surface. Let us put

h=the heat of any body B;

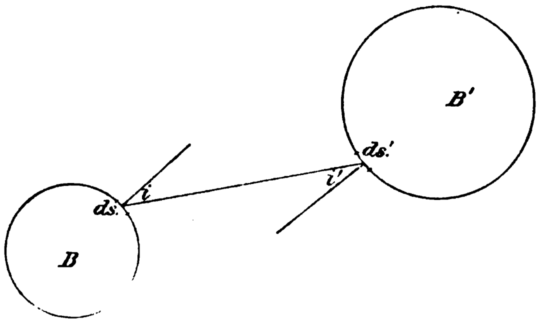
θ=its temperature;

ds=a differential element of its surface;

$d\sigma$=an element of a solid angle with vertex at ds;

i=the angle of emission with reference to the normal to the surface.

Fig. 1.



The rate of radiation, then, from any differential element of the surface of the body B regarded as a point, within the element $d\sigma$ of a solid angle, is expressed by

$$(1) \qquad \frac{dh}{dt}=K\rho\cos i\varphi(i)\varphi(\theta)dsd\sigma$$

in which K is a constant depending upon the absolute radiating power of a lamp-black surface of temperature $\theta=0$ in a direction normal to the surface, the radiating power of such a surface being a maximum; ρ is the relative radiating power of any other kind of surface—that is, radiating power relative to a lamp-black surface; $\varphi(i)$ is a function of i which generally differs but little from unity, causing the radiating power to vary a little from the law of $\cos i$ called the *law of cosines;* and $\varphi(\theta)$ is a function of the temperature to be determined from experiment.

In the case of a lamp-black surface we have $\rho=1$, and as in this case the radiation in the different directions follows the law of $\cos i$ we have in this case $\varphi(i)=1$ also.

For any other body B', distinguishing the quantities h, θ, ds, &c., by an accent, we have

$$\frac{dh'}{dt}=K\rho'\cos i'\varphi(i')\varphi(\theta')ds'\,d\sigma'$$

for the rate by which heat is radiated from ds' within the solid angle $d\sigma'$. Expressing ds' and $d\sigma'$ in terms of $d\sigma$ and ds, we have, putting f=the distance between ds and ds'

$$(2) \qquad ds'=f^2\frac{d\sigma}{\cos i'} \qquad\qquad d\sigma'=\frac{ds\cos i}{f^2}$$

A lamp-black surface absorbs all the heat falling upon it, while others absorb only a part and reflect the balance if they are adiathermanous; but if they are diathermanous they reflect and transmit all which is not absorbed. A lamp-black surface, therefore, has a maximum absorbing power, and in other bodies it is less and is also a function of the angle of incidence i. The relative absorbing power may therefore be expressed for any given direction by $\alpha\varphi'(i)$. We shall therefore have for the rate with which the heat is received from ds' and absorbed by ds

$$(3)\qquad \left(\frac{dh'}{dt}\right)=K\varphi(\theta')\rho'\varphi(i')\cos i'\alpha\varphi'(i)ds'\,d\sigma'$$

Putting for $d\sigma'$ its value in (2), this becomes

$$\left(\frac{dh'}{dt}\right)=K\varphi(\theta')\rho'\cos i'\varphi(i')\frac{\cos i}{f^2}\alpha\varphi'(i)ds'\,ds$$

In like manner we have for the rate with which heat is received from ds and absorbed by ds', by inverting the accented and unaccented letters in the preceding expression required in an inversion of the case

$$\left(\frac{dh}{dt}\right)=K\varphi(\theta)\rho\cos i\varphi(i)\frac{\cos i'}{f^2}\alpha'\varphi'(i')ds\,ds'$$

If in these last two expressions we put

$$(4)\qquad \theta=\theta' \qquad\qquad \rho\varphi(i)=\alpha\varphi'(i) \qquad\qquad \rho'\varphi(i')\doteq\alpha\varphi'(i')$$

they become the same. The first of these conditions expresses equality of temperature of the two bodies, and the other two equality in the radiating and absorbing powers of the two bodies for equal angles of emission and incidence, expressed in both cases by i or i'. Hence if two bodies in the same vicinity have the same temperature, and the absorbing and radiating powers of each one are the same, though differing in the two bodies, *the heat received by any differential element of the surface* ds *of the one body from any differential element ds' of the other and absorbed is precisely equal to the heat received by* ds' *from ds and absorbed.* As it is only absorbed heat which affects temperature, the equality of the temperatures are therefore not affected by any interchange of heat between the two bodies where the radiating and absorbing powers are equal for all angles of emission and incidence.

70. It is usually said that radiation and absorption are equal, and it can be shown that this is the case where the temperatures of the source of heat and of the body absorbing and radiating the heat are the same, but it does not seem to be true as a general proposition. According to Knoblauch[53] the radiating power of a body is the same be the calorific rays with which it is heated of ever so different kinds, and this seems to be generally admitted. But the absorbing power seems to depend upon the temperature of the source of heat. Since the absorbed heat is simply the difference between the incident and reflected heat, this is best shown from observation of the reflecting powers. The following

table contains the results of experiments made by MM. de la Provostaye and Desains on the reflecting power of several of the metals :[54]

Metals.	Locatelli lamp.	Solar rays.	Alcohol lamp.
Polished silver plate	0.97	0.92	....
Gold	0.95	0.87	....
Brass and copper	0.93	....	0.945
Metal of mirrors, polished	0.86	0.64	....
Pewter	0.85	0.60	0.86
Polished platinum	0.80	....	....
Steel	0.83	0.60	0.83
Zinc	0.84	....	....
Iron	0.77	....	....
Cast iron	0.74, 0.75	....	....

The complements to unity of these several reflecting powers are the absorbing powers. Hence it is seen that the absorbing powers of the metals are much greater for the heat of the solar rays than for the heat of a Locatelli or alcohol lamp.

With regard to different angles of incidence it seems, from the results of the same experimenters, that the reflecting, and consequently the radiating, power of metals is the same for all angles of incidence up to 70°, after which the reflecting power diminishes, and at about 80° it is nearly 0.94 of its value at small angles of incidence. From the reflecting powers, the same experimenters found that the radiating power of glass is 0.90, with an incidence of 0°; 0.836 when the angle is 60°; 0.7504 for 70°; 0.653 for 75°; and 0.5544 for 80°. These are the values in this case of $\rho\varphi(i)$ in (4), and hence $\rho \cos i \; \varphi(i)$ is the radiating power for an angle of emission i, and as the values of $\rho\varphi(i)$ above decrease with increase of angle of emission, the radiating powers of glass decrease with increase of i more than in the case of a lamp-black surface, in which it is, by Lambert's law, as $\cos i$.

Magnus[59] obtained 8.4 for the per cent. of heat reflected by a glass plate which was radiated from a silver plate, imperfectly coated with lamp-black, at an angle of incidence of 45°. The temperature of the source was 150° C. This makes the radiating power of glass considerably greater than it is by the experiments of MM. de la Provostaye and Desains.

In diathermic bodies the absorption is diminished by the amount of heat transmitted through the body as well as by that reflected. In this case, therefore, the relation between radiation and absorption is often very different from what it is in the case of adiathermic bodies. In the latter we have seen that the relation depends in some measure upon the temperature of the source from which the heat comes, the absorption for several of the metals being greater in the case of the solar rays than in that of the Locatelli lamp, while the radiating power is the same in both, though for equal temperatures between the body receiving the rays and the body from which the rays come, the radiating and absorbing powers are equal, and sensibly so in the case of any ordinary differences of temperature. But in adiathermic bodies, while the radiated heat is the same for the same body whatever the source from which it receives heat, the absorbed heat depends upon the size of the body and its transmissive power for heat. For low-temperature sources of radiation the transmitted heat is in general insensible. This is shown by the experiments of Melloni[27]. Seven plates of glass of different degrees of thickness, submitted to the action of four sorts of calorific rays, gave the following transmissions:

Thickness of plate.	Per cent. of transmissions.			
	Locatelli lamp.	Incandescent platina.	Blackened copper, 390° C.	Blackened copper, 100° C.
mm				
6.07	77	54	34	12
0.5	54	37	12	1
1.0	56	31	9	0
2.0	41	25	7	0
4.0	37	20	5	0
6.0	35	18	4	0
8.0	33.5	17	3.4	0

The temperatures of the first two are unknown, but the first is greater than the second, and the second than the third. Now, a single glance at the table shows that the number of rays transmitted through the same plate differs with different sources of calorific rays, and that the transmissibility seems to increase with the temperature of the source. This, however, does not seem to be a general law, for Melloni found two exceptions. Pure rock salt is penetrated by heat, according to Melloni, from every source in a uniform manner, and prepared rock salt, according to Melloni and Forbes, is penetrated by heat in a degree which decreases as the temperature of the source is increased[53].

The following are the per cents. of transmission through a plate of several diathermanous bodies of 2.6^{cm} in thickness for the several different sources of heat, as given by Melloni:[55]

Substance.	Locatelli lamp.	Incandescent platina.	Blackened copper.	
			390° C.	100° C.
Rock salt (clear, colorless)	92	92	92	92
Rock salt (clear, dull)	65	65	65	65
Fluate of lime (clear, colorless)	78	69	42	33
Iceland spar (clear, colorless)	39	28	6	0
Mirror glass (clear, colorless)	39	24	6	0
Ice, very pure (clear, colorless)	6	0	0	0

In all these cases, except that of rock salt, the transmissivity increases with the temperature of the source of radiation. In the case of the solar rays, on account of the high temperature of the sun, there is perhaps no diathermic body through which the heat is not transmitted with greater facility than that from terrestrial sources of comparatively low temperature. The reflected rays of the sun have also very nearly, if not quite, the same high penetrative power, the quality of the rays not being much changed by reflection, so that the sun's reflected heat penetrates glass with apparently the same facility as the rays before reflection, while that radiated from terrestrial objects of low temperature, and even by the flame of combustion, is almost wholly cut off by a thin plate. It is, however, well established that the transmissivity of the same body differs very much with the source of the heat, and consequently that in diathermic bodies the relation between radiation and absorption differs in different bodies, the former being generally greater than the latter, and most probably always so where the source of heat is the sun.

In the case of a plate of glass one millimeter in thickness, allowing 10 per cent. for reflection, it is seen, from the first of the preceding tables of results, that 44 per cent. is absorbed of the heat from a Locatelli lamp, 59 of that from incandescent platina, 81 from blackened copper of 390° C., and 90 of that from blackened copper of 100° C., while the radiation is the same in all. If we take a plate four millimeters in thickness the amount of absorption is considerably greater for all sources of radiation, while the radiation is nearly the same, this depending merely upon the extent of surface. Hence the difference between radiation and absorption differs both with the source of the radiation and the thickness of the plate.

The effect of the temperature of source upon the diathermancy of glass plates is shown by the experiments of Schneebeli.[50] The absorption coefficient k was reckoned from the formula

$$J=J_0 e^{-kx} \text{ or } k=\frac{\log\frac{J_0}{J}}{x}$$

in which

x = the thickness of the absorbing plate;
J_0 = the intensity of the ray falling upon the plate;
J = the intensity of the transmitted ray.

He obtained the following values of k for the different temperatures of the radiating body in the first column, the thickness of glass plate being 1.75^{mm}.

Temperature of body.	Value of k.
100° C.	2.4
250	1.47
About 1,000	0.42

It is seen when the temperature of the radiating body was 1,000° the absorption coefficient was only about one-sixth as much as when its temperature was only 100°, and hence the diathermancy of the glass for heat from a high temperature source was much greater. The constant k includes reflection and absorption, but there was most probably but little difference in the reflections in the several cases, and therefore the differences are mostly due to different absorptivities.

Law and rate of the cooling of bodies in vacuo.

71. If we put

H = the difference between the heat radiated by B and that received from B' by B and absorbed;

C = the capacity* of B for heat;

we shall have

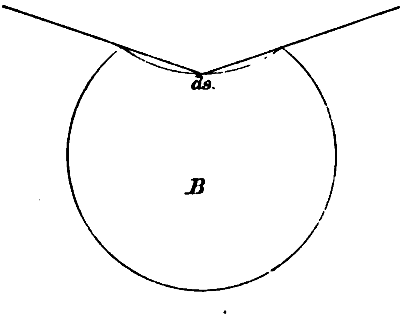
$$C\frac{d\theta}{dt}=\frac{dH}{dt} \tag{5}$$

The integration of (1) with regard to ds and $d\sigma$ gives the rate with which B loses heat by radiation. In like manner the integration of (3) with regard to ds' and $d\sigma'$ gives the rate with which heat is received from B' by B and absorbed. We therefore have

$$\begin{aligned}\frac{dH}{dt}=&-K\varphi(\theta)\int\rho\varphi(i)\cos i\,d\sigma\int ds\\&+K\varphi(\theta')\int\alpha\varphi'(i)\,d\sigma'\int\rho'\varphi(i')\cos i'\,ds'\end{aligned} \tag{6}$$

The first term of the second member, expressing the rate with which B radiates heat in all directions from its surface, must be integrated with regard to $d\sigma$ so as to include in the integral the whole solid angle in which the radiations from ds are contained. For a spherical body or any other which does not have depressions in its surface, this solid angle is a hemisphere. But if any element ds of the surface of B is in a depression, as in Fig. 2, then this solid angle of integration is less than a hemisphere, and consequently the rate of radiation from ds is less than for an element anywhere on a convex part of the surface where radiation takes place in all directions above a tangent plane. For every convex part of the surface the rate of radiation is the same, so that where there are no depressions the radiation is in proportion to the surface, whatever the

FIG. 2



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[*] As used here the heat required to raise the temperature of the body from 0° to 1° C.; the *specific heat capacity*, or *specific heat* simply, being used to denote the heat required to raise the temperature of one unit of weight through the same range of temperature.

figure of the body may be, but where there are depressions the rate of cooling is less than in case of a body of the same extent of surface which is all convex. The integration with regard to ds must include the whole surface of B. In the last term of the expression of (6) the integral with regard to $d\sigma'$ must be taken for each element ds', through the solid angle subtended by the body B, and with regard to ds' so as to include all the elements of surface which emit rays to B.

In the case of a body of which all parts of the surface are convex the integral $\int \rho\varphi(i) \cos i d\sigma$ has to be taken for each element ds through a solid angle of the extent of a hemisphere and consequently this integral is a constant for each of those elements, and we have

$$\int \rho\varphi(i) \cos i d\sigma \int ds = s \int \rho\varphi\ (i) \cos i d\sigma$$

In the case of a lamp-black surface we have $\rho=1$ and $\varphi\ (i)=1$. Hence if we put

r=the radiating power of a convex surface relative to that of a convex lamp-black surface,

we shall have

$$(7) \qquad r=\frac{\int \rho\varphi(i) \cos i d\sigma}{\int \cos i d\sigma}=\frac{\int \rho\varphi(i) \cos i d\sigma}{\pi}$$

From this and the preceding equation we get

$$(8) \qquad \int \rho\varphi(i) \cos i d\sigma \int ds = \pi r s$$

in which s is the whole surface of B.

The integration in the last term of the expression of (6) cannot be accurately obtained in many cases, but approximately in all.

Let us put

u=the value of i' for the ray passing from ds' toward the center of B, supposed here to be spherical;
ω=the angle between the ray of which the angle of emission is u and that of which the angle of emission is i';
ω'=the extreme value of ω where B is a sphere;
D=the distance between ds' and the center of B;
$d\psi$=the differential element of a solid angle subtended by ds' as seen from the center of B;

We shall then have $\cos u ds' = D^2 d\psi$ and by means of this we can put

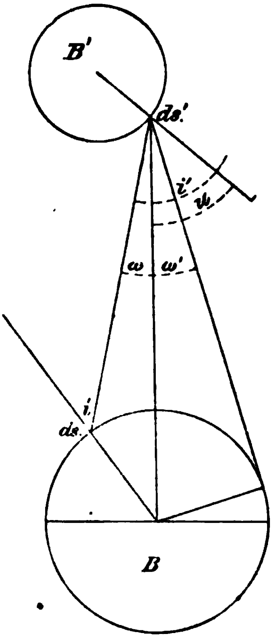
$$(9) \quad \int \alpha\varphi'(i) d\sigma' \int \rho'\varphi(i') \cos i' ds' = ar' D^2 \int \cos \omega d\sigma' \int d\psi \,.\, \frac{\varphi'(i)\varphi(i')\cos i'}{\cos u \cos \omega}$$

in which

a=the mean relative absorbing power of B for all angles of incidence,
r'=the mean relative radiating power for all angles of emission.

It is seen from Fig. 3 that the ray of which the angle of emission is i' may be in any direction around that of which the angle of emission is u, and consequently the former is less on the one side and greater on the other than u, and cos ω differs but little from unity unless the body B is such and so placed as to subtend a very large solid angle as seen from ds'. In the case of a lamp-black surface of B and B' both $\varphi'(i)$ and $\varphi(i')$ become unity, and for any kind of surface they mostly vary so little with different angles of incidence and emission that they may be regarded as very nearly constant within the ranges of the integrations. We shall therefore incur very little error in the integrations by regarding the last factor in the second member above as a constant and equal to its mean value, obtained by putting $i'=u$.

FIG. 3.



In the integration of $\int \cos \omega d\sigma'$ it will be necessary to express $d\sigma'$ in terms of $d\omega$. For this purpose we must put, where B is a sphere, $d\sigma'=2\pi \sin \omega d\omega$. With this we get

$$\int \cos \omega d\sigma' = \int 2\pi \sin \omega \cos \omega d\omega = 2\pi \int \sin \omega \, d \sin \omega_1 = \pi \sin^2 \omega'$$

in which ω' is the limiting value of ω including the whole body B. Let us put

R=the radius of B when a sphere;
S=the sectional area of a great circle of B;

we shall then have, as is readily seen from Fig. 3, $D \sin \omega' = R$, and hence, by means of the preceding equation,

$$(10) \qquad D^2 \int \cos \omega \, d\sigma' = \pi D^2 \sin^2 \omega' = \pi R^2 = S$$

Hence this is a constant for all values of D. By means of this we get from (9)

$$(11) \qquad \int \alpha \varphi'(i) d\sigma' \int \rho' \varphi(i') \cos i' ds' = ar' S \psi Q$$

in which ψ is the solid angle subtended by the body B', and in which

$$(12) \qquad Q = \varphi \left(\frac{\varphi(i) \varphi(i') \cos i'}{\cos u \cos \omega} \right)$$

is a function which in general does not differ sensibly from unity unless the angles i', u, and ω are very large. The integration with regard to $d\psi$ is entirely independent of the figure of the body B', and consequently (11) holds for a body of any shape.

As the second member of (11) is an expression of the heat received from B' by B and absorbed, if we put $\alpha=1$ it becomes the expression of the heat received from B' by B, since when $\alpha=1$ all is absorbed which is received. Hence *the heat received by a spherical body from any body B' of any figure is proportional to the solid angle subtended by B' as seen from the center of B*, except so far as it is slightly changed by the varying values of Q in (12), with change of distance, or change of figure of the body upon which the values of i', u, and ω for the different elements of surface ds' depend.

Where B' is a spherical body the value of Q in (12) differs most from unity for the elements of ds' on the outer part of the solid angle or disk as seen from the center of B, since for these elements the values of i' and u in (12) are large, and where the body B is such also as to subtend a large angle as seen from ds', in which case the value of ω and the difference between i and u may be large, and the value of Q a little less than unity, where the integration extends to these border elements of surface ds' only.

The function $\varphi'(i)$ depends upon different absorbing powers of the surface of B for different angles of incidence, but this difference is generally very small and entirely vanishes in a lamp-black surface, where all are absorbed and $\varphi'(i)$ becomes unity. The function $\varphi(i')$ depends upon the different radiating powers of the surface of B' for different angles of emission i, the effect of which would be that a little more or less heat is received from the central part of the disk of a spherical body than from a part of the exterior part of the disk subtending the same solid angle.

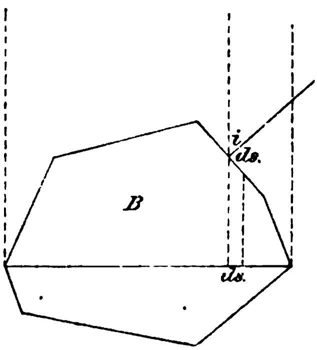
Where the element ds' is normal to the direction of the center of B, Fig. 3, we have $u=0$, and $\omega=i'$, and also $\varphi(i')=1$, very nearly, since in this case the angles of emission are all small unless B, as seen from ds', subtends a large solid angle. In this case, therefore, the value of Q is very nearly unity, and exactly so where the surfaces of B and B' are lamp-black surfaces.

72. By means of (2) we can put

$$(13)\qquad \int \alpha\varphi'(i)d\sigma' \int \rho'\varphi(i')\cos i'ds' = \int \alpha\varphi'(i)\cos i\,ds \int \rho'\varphi(i')d\sigma$$

in which $d\sigma$ is the solid angle corresponding to any element of ds' of the surface of the body B'. Where the two bodies are so far apart that the solid angle of each one as seen from the other is very small, the rays passing from all parts of B' to B may be regarded as parallel, and the solid angle subtended by B' as seen from B may be regarded as the same for each element ds of the surface of B. Hence we can in this case put $d\sigma=d\psi$. Letting S here represent the section of B, normal to the direction of the parallel rays, which intercepts the greatest number of rays, Fig. 4, we shall have in this

FIG. 4.



case, whatever may be the figure of B, $\cos i ds = dS$. If we, therefore, put

a=the relative absorbing power of the body for parallel rays falling upon it at the several angles of incidence,

we shall have

$$(14) \qquad a = \frac{\int \alpha\varphi'(i) \cos i ds}{\int \cos i ds} = \frac{\int \alpha\varphi'(i) \cos i ds}{S}$$

By means of this, and the relation of $d\sigma = d\psi$, (13) gives, where $\varphi(i')=1$.

$$(15) \qquad \int \alpha\varphi'(i) d\sigma' \int \rho' \varphi(i') \cos i ds' = a\rho' S\psi$$

which is the same result as obtained in (11) for a spherical body in the case of lamp-black surfaces and with B' at an infinite distance, for in this case we have, in (12), $Q=1$.

73. By means of (8) and (11) we now get from (6)

$$(16) \qquad -\frac{dH}{dt} = K[\pi r s \varphi(\theta) - a r' S \psi Q \varphi(\theta')]$$

in which Q can generally be taken as unity without incurring any sensible error.

In the case in which the rays can be regarded as parallel, using (15) instead of (11), we get, where $\varphi'(i)=1$, as it is with a lamp-black surface, and very nearly in all cases, the same expression of (16), except that in this case we have $Q=1$ precisely, and S is the section of the body intercepting the greatest amount of the rays from B'. In this case the body B may have any figure whatever, unless there are concavities on its surface, and the result is the same.

If there are n bodies such as B', then the effect of each one upon the rate of cooling of B is the same as if the others were not present, unless they are so situated that the different solid angles which they subtend, or their disks as seen from B, interfere or overlap. In this case the solid angle ψ for each body must comprise the visible part only, unless the intervening bodies are diathermic. We, therefore, get, instead of (16), in the case of several bodies radiating heat to B, by regarding $Q=1$,

$$(17) \qquad -\frac{d\theta}{dt} = \frac{K}{C}[\pi r s \varphi(\theta) - \Sigma_n a r' S \psi \varphi(\theta')]$$

in which the quantities r', ψ, and θ' under the sign Σ_n may have different values for each of the several bodies. The value of S will also be different for each of the different bodies unless B is spherical, in which case the section S intercepting the rays is the same for all directions of B'; but this would not be so for a body of any other form, such as is represented in Fig. 4.

The first term in this expression is the rate by which a single body cools in space where it receives no heat from any other body. This rate is less in large than in small bodies of the same figure, since C in the denominator is as the cubic dimension of the cooling body, while S is the superficial dimension. It is seen that the presence of B' diminishes this rate, and the more so the greater are a and S, the absorbing power of B and its maximum sectional area where B is spherical. It also depends upon, and is proportional to, the radiating power r' of B' and the solid angle ψ which it subtends, as seen from B.

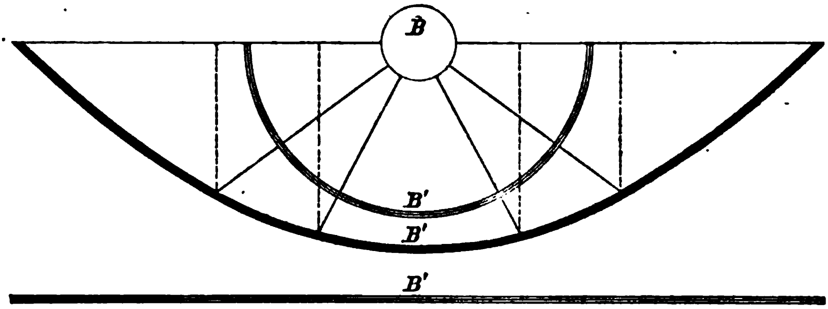
Where B', Fig. 5, is an infinite plane statum, as seen from B, it subtends a solid angle of a hemisphere, and hence in this case we must put in (16), $\psi=2\pi$, and, where B is spherical, $s=4\ S=4R^2\pi$. With these values in (16), neglecting Q, we get very nearly

$$(18) \qquad -\frac{d\theta}{dt}=4R^2\pi^2\frac{K}{C}[r\varphi(\theta)-\tfrac{1}{2}ar'\varphi\theta']$$

But this is applicable in the case only in which B is spherical, since for any other figure S is not the same for all directions of the elements ds' of the surface of the body B'.

The same expression is applicable to any body which forms a hemispherical inclosure, whether the surface which radiates to B is that of an infinite plane, or is parabolic, hemispheric, or that of any other figure, since the only condition required in (18) is that it shall subtend a solid angle of a hemisphere.

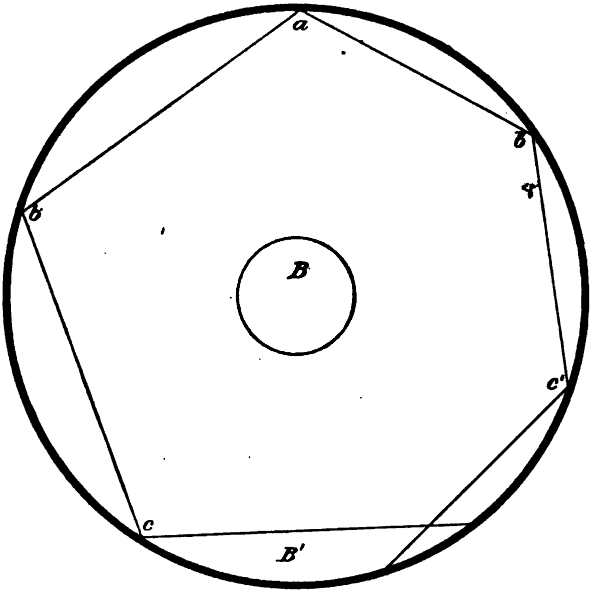
FIG. 5.



In the preceding expressions the radiated heat only of B' has been taken into account. In (16) a single body, even when small, might have a form which would reflect some heat back to B, and in (17) where there are supposed to be a number of bodies from which B receives heat, there might be considerable heat reflected to B if the bodies B' are reflecting bodies. Whenever this is the case it is necessary to take the reflected heat which reaches B into account, whether this is its own radiated heat reflected back or that of some other of the bodies B', or from a different part of B' when there is only one. In the case of the infinite plane, or of the body *B′*, with a parabolic reflecting surface, no heat reflected from *B′* reaches *B*, either of its own radiated heat or that from some other part of *B′*; but in the case of the hemispherical inclosure it nearly all comes directly back to *B*, and all would if *B* were a mere point. No heat, however, even in this case, which is radiated from any part of *B′* is reflected by another part of *B′* to *B*.

74. In the case in which *B′* forms a *complete inclosure*, whether spherical or otherwise, it is very important to take into account the reflections, if it is a reflecting body; for in this case no heat which is radiated from each point in all directions is reflected out so as not to return, as in the preceding cases, but it is reflected and re-reflected in all directions from side to side, except the part which is absorbed at each reflection, until the radiated and reflected heat from any part of the inclosure is exactly equal to that which would be radiated by a lamp-black surface, however small the radiating power of the inclosure may be.

FIG. 6.



The radiating power of any element of the surface of *B′* for rays emitted in all directions, as from *a* to *b*, *b′*, &c., is expressed by *r* in (7) in a function of the angles of emission *i*. The rays coming in from all directions to *b*, *b′*, and *c*, *c′*, &c., whatever the figure of the inclosure, are reflected, likewise, in all directions, and if the radiating and absorbing powers are the same the part absorbed at *b*, *b′*, and *c*, *c′*, &c., is rr, and the part reflected $r(1-r)$. At the second points of reflection the $(1-r)$ part of this is reflected, and hence the part of the radiation from *a* which is reflected at the second reflection is $r(1-r)(1-r)$. At the third reflection it is $r(1-r)(1-r)(1-r)$, and so on. Hence we shall have for the sum of the heat radiated from a and the sum of all the reflections,

$$(19) \qquad r+r(1-r)+r(1-r)^2+r(1-r)^3+\&c.=\frac{r}{1-(1-r)}=1$$

which is the radiating power of a lamp-black surface since the radiating power here is expressed relatively to that of such a surface. This holds, whatever may be the law of radiation and reflection, with regard to the angles of incidence and emission with regard to i, whether as cos i or any other function, cos $i\varphi(i)$.

In the numerous reflections and re-reflections some of the rays fall upon, and are reflected by B, and also the rays radiated by B are reflected by the inclosure B', and if B has the same radiating and reflecting powers as the inclosure, and the same temperature, the result is the same, and for every part of the inclosure and of B the sum of the radiated and reflected heat is equal to that which would be radiated by a lamp-black surface, assumed here as unity. If the radiating power of B, or even some part of the inclosure B', were not r but r', thus some of the factors in (17) would be $(1-r')$ instead of $(1-r)$, but the final result would be the same. Suppose we have $(1-r')$ at the third reflection and put

$$r+r(1-r)+r(1-r)^2+r(1-r)^2(1-r')=a$$

If we then suppose the law to continue as before, we shall have

$$a+a(1-r)+a(1-r)^2+a(1-r)^3+\&c.=1$$

If there was another deviation from the regular law, as $(1-r'')$, in the second series of reflections, it would be shown in the same manner that the final result must still be unity; and so, if there were any number of such deviations from $(1-r)$. The only effect upon the final result would be a little acceleration or retardation in arriving at it by the several approximations; but finally the radiation from any point of either B, or the inclosure B', together with the sum of all the reflections is equal to unity—that is, to the radiation from a lamp-black surface.

In the case of a complete inclosure, therefore, we must put, in (6), $\rho'\varphi(i')=1$ since the heat emitted from all parts of the inclosure by radiation and reflection together is the same as that radiated by a surface of maximum radiating power.

In this case, therefore, we can put in (13)

$$\int a\varphi'(i)d\sigma'\int\rho'\varphi'(i')\cos i'ds'=\int ds\int a\varphi'(i)\cos i d\sigma$$

In this case the last integral of the second member becomes the same for each element ds of the surface of B, provided there are no depressions in the surface, since the integration with regard to $d\sigma$, for all parts of the surface of B, must extend through the same solid angle of a hemisphere, for the rays come in to ds from the inclosure in all directions above a tangent plane at that point. We therefore get

$$\int \alpha\varphi'(i)d'\sigma \int \rho'\varphi(i') \cos i'ds' = \pi as \tag{20}$$

in which

$$a = \frac{\int \alpha\varphi'(i) \cos i d\sigma}{\int \cos i d\sigma} = \frac{\int \alpha\varphi'(i) \cos i d\sigma}{\pi} \tag{21}$$

is the absorbing power of the surface of B (relative to that of a lamp-black surface) for rays of all angles of incidence.

This expression of a is the same as that of r in (7), if we put $\alpha\varphi'(i) = \rho\varphi(i)$—that is, the absorbing equal to the radiating power for each separate angle of emission or incidence. It, however, differs a little from that of a in (14), unless B in that case is spherical or has a maximum absorbing power, since in (14) the rays do not have all angles of incidence where B is not spherical.

By means of (5), (8), and (20) we now get from (6)

$$-\frac{d\theta}{dt} = \frac{SK\pi}{C}[r\varphi(\theta) - a\varphi(\theta')] \tag{22}$$

As both (8) and (20) are independent of the figure of the body B, except that it must not have depressions in its surface, and likewise of the figure of the inclosure, this expression applies to a body B of any figure, with the exception stated, and to any kind of a complete inclosure with any radiating power whatever.

The preceding expression for a special and comparatively simple case, where B is spherical, is almost exactly deducible from that of (16) for the general case of only partial inclosures, and exactly in case B has a lamp-black surface; for in the case of a complete inclosure we have, in (16), $\psi = 4\pi$ and $s = 4s$, and in case of a lamp-black surface $\rho' = 1$, and, as is seen from § 65, $Q = 1$. With these values (16) becomes the same as (22).

The expression of (22) is substantially the same as the result obtained by Pouillet[56] in the case of a spherical body within a complete spherical inclosure, if we assume the absorbing and the radiating power of the body to be equal. But it is important and interesting to have the rates and laws of cooling for the more general conditions of partial inclosures, and bodies of various figures, since from these the equations of condition are formed for determining the temperature under these various circumstances. In this more general treatment of the subject it is necessary to consider the law of relative radiation and absorption with reference to the angles of emission and incidence, since these are now known to differ in different bodies with reflecting surfaces, as has been seen in § 70. The effect, however, arising from this consideration, is in general small. In the simple case of a spherical body in a complete spherical inclosure it is only necessary to have the mean radiation and absorption for rays emitted and received from all directions within a hemisphere, as given in (7) and (21), whether the radiation and absorption for all angles of emission or incidence are the same or not.

75. So far no account has been taken of the heat reflected either in the case of partial or complete inclosures, except of the reflected heat in the latter case corresponding to the radiation of B when it has the same temperature as the inclosure, in which case we have seen the heat emitted from the body B by radiation and reflection together is the same as that which it would radiate with a maximum radiating power, assuming that the radiating and absorbing powers are equal. This, however, is not the case where B has a greater temperature and so a greater absolute radiating power than the inclosure. In this case the part of the heat arising from this increased radiation which is reflected by the inclosure back to B tends to diminish the rate of cooling. If the body B therefore were situated in the center of a perfectly spherical inclosure the part of the whole heat radiated by B which would be reflected back to B would be $(1-r')$, r' being the radiating power of the inclosure. The rate of cooling, therefore, with B thus centrally located in the inclosure would be less than it would be in an inclosure with maximum radiating and no reflecting power. If, however, the body B is so placed, as it may, that none of the reflected rays come back to it, except from irregular reflection, then the rate of cooling is the same for reflecting and non-reflecting inclosures.

76. In all the preceding expressions of the rate of cooling under the various circumstances, there is the function $\varphi(\theta)$, a function depending upon the temperature, which can only be determined by experiment and observation. This has been done by Dulong and Petit[37] in the case of a large thermometer bulb of glass, 6 centimeters in diameter, filled with mercury and placed *in vacuo* within a spherical inclosure with lampblack surface. The rates of cooling were observed in the inclosure at the several temperatures of 0°, 20°, 40°, 60°, and 80° C. The observed rates of each of these temperatures of the inclosure were found to be well represented, through a range of 240° C., by an empirical formula of the following form:

$$(23) \qquad -\frac{d\theta}{dt}=m\,(\mu^{\theta}-\mu^{\theta'})=m\mu^{\theta'}(\mu^{\theta-\theta'}-1)$$

in which

$$(24) \qquad m=2.037 \qquad \mu=1.0077$$

After values of these constants m and μ had been determined, which represented the observations satisfactorily, still other observations were made and compared with the values computed from the formula with these values of the constants.

It is in accordance with universal experience that a body, when placed within a complete inclosure, cools down to the temperature of the inclosure. At this point the first member of (22) vanishes, and in order to satisfy the equation for $\theta=\theta'$ it is necessary to have $r=a$—that is, the relative radiating and absorbing powers equal. The truth of this principle, therefore, depends upon observation, as well as the law of (23). But r and a are simply the average for all angles of emission and incidence, as expressed in (7) and (21), when they have a range of a whole hemisphere; but these expressions and the temperature of the body and inclosure could not be equal under all the varying circumstances of figure of both the body and the inclosure, unless each element of the surface of the one should receive from each element of the other and absorb as much heat as is returned to the other and absorbed, which we have seen (4) requires that radiation and absorption must be the same, not only for all angles of emission and incidence taken together, but separately.

Putting $r=a$, and comparing (23) with (22), since the conditions of (20) correspond with those of the experiments, we get

$$(25)\qquad \frac{rsK\pi}{C}=m=2.037 \qquad \varphi(\theta)=\mu^{\theta} \qquad \varphi(\theta')=\mu^{\theta'}$$

Since, however, the experiments did not extend down to temperatures below freezing, this last equation cannot be safely extended to very low temperatures, for the law, being merely empiric, may not hold, even approximately, if applied far beyond either of the limits of the range of temperature for which the experiments were made.

The constant K, as it has been used, expresses the rate of radiation of a unit of surface of maximum radiating power through a unit of solid angle as measured by the surface of a sphere of unit radius, if the intensity of radiation were the same for all angles of emission. $K\pi$, therefore, expresses the rate of radiation of a unit of surface in all directions within a hemisphere. If we therefore put, as Pouillet has done,

B=the rate with which heat is radiated from a unit of surface of maximum radiating power,

we shall have by (25)

$$(26)\qquad K\pi=B=\frac{2.037\ C}{rs}\ .$$

With the capacity for heat C known from the weights of mercury and glass in the cooling body, and their specific heats, and with the value of s, the surface of the body, and an assumed value of $r=0.8$ for glass, Pouillet[56] determined the value of B to be 1.146. The units of surface, temperature, and time being respectively the centimeter, degree centigrade, and minute, the rate is in calories per minute.

This value of B, however, must be regarded as only approximate, since there was an uncertainty, both with regard to the size of the bulb used by Dulong and Petit, and also with regard to the exact radiating power of glass. The experimenters had in view the *law* simply of cooling and not the absolute rate of losing heat, and hence the exact dimension of the bulb was not needed for their purpose, and therefore was perhaps not very accurately determined. With regard also to the true radiating power of glass there is still considerable uncertainty, but it is now generally supposed to be greater than 0.8, which Pouillet used. A better determination of this important constant ought to be made from experiments with a body of maximum radiating power and accurately-determined extent of surface and capacity for heat.

77. The value of r may be determined from (7) by means of mechanical quadration through a hemisphere, with the values of $\rho\varphi(i)$ given for glass in § 70. Putting in (7) $d\sigma = 2\pi \sin i \, di$, we get

$$(27) \qquad r = 2 \int \rho\varphi\,(i) \cos i \sin i \,\delta i = \int \rho\varphi\,(i) \sin 2\,i\,\delta i$$

From the several values given in § 70 we get for intervals of 5°, with considerable probability, the values given in the second column of the following table. For these intervals we have

$$\delta i = \frac{5\pi}{180} = \frac{1}{36}\pi = .0873$$

With this value of δi we get the third column in the following table:

i	$\rho\phi i$	$\rho\phi(i) \sin 2\,i\,\delta i$
°		
2.5	.899	.0068
7.5	.897	.0200
12.5	.895	.0330
17.5	.892	.0446
22.5	.888	.0548
27.5	.883	.0632
32.5	.878	.0696
37.5	.874	.0736
42.5	.869	.0756
47.5	.863	.0752
52.5	.855	.0722
57.5	.843	.0668
62.5	.818	.0588
67.5	.776	.0480
72.5	.703	.0350
77.5	.604	.0182
82.5	.502	.0114
87.5	.395	.0030
		$r = .8298$

The sum of the numbers in the last column of this table gives approximately the value of r. The method of quadrature is sufficiently accurate where there is so much uncertainty in the data. This gives the value of r very nearly 0.83, which is greater than the value used by Pouillet. According to Magnus, § 70, it would be still greater. With these greater values (26) would give a value of B a little less than 1.146; but on account of the uncertainty of the data throughout we cannot do better than to retain for the present Pouillet's value.

It is seen in all the preceding expressions that the radiating power r of the cooling body, as well as that of the inclosure r', is an important element in the conditions determining the rate of cooling and, as will be shown, the temperature of the body. This has been determined by Leslie and others for a number of substances, but the results which they have obtained must be regarded as only approximate. They are as follows:

Leslie.	
Lampblack	100
White lead	100
Writing paper	98
Ordinary white glass	90
Isinglass	80
Tarnished lead	45
Mercury	20
Polished lead	19
Polished iron	15
Tin, gold, silver, and copper	12

MM. de la Provostaye and Desains.	
Lampblack	100
Platinum, laminated	10.80
Burnished platinum	9.50
Silver deposited chemically on copper	5.36
Copper foil	4.90
Gold leaf	4.28
Pure silver, laminated	3.00
Pure silver, burnished	2.50
Pure silver deposited chemically and burnished	2.25

78. Putting in the general expression of (16) for $\varphi(\theta)$, $\varphi(\theta')$ and $k\pi$ their values in (25) and (26), we get, when the body B is spherical and consequently $s=4S$, very nearly in all cases

$$(28)\qquad -C\frac{d\theta}{dt}=-\frac{dH}{dt}=BS(r\mu^{\theta}-ar'e\mu^{\theta'})$$

in which

$$(29)\qquad e=\frac{\psi}{4\pi}$$

is the proportion of solid angle subtended by the partial inclosure in comparison with a complete inclosure. In some cases in which the body B', or the partial inclosure, is very near to B, we have seen that Q may vary sensibly from unity, and in such cases (28) would be only approximate. In case of a complete inclosure, or partial inclosure with lampblack surface, we have $r'=1$.

In the application of this general expression to the cases of § 73 we should have $e=\frac{1}{2}$, but in that of a complete inclosure we must put $e=1$. In the latter case also r' becomes unity for all kinds of surfaces, since the radiated and reflected heat together become equal to that radiated by a surface of maximum radiating power.

Although in adiathermanous bodies we have $r=a$ where $\theta=\theta'$, yet it is seen in § 70 that when the source of heat has a high temperature, as in the case of the sun, a may change while r does not, and hence in such cases we do not have $r=a$. In the case also in which B is diathermanous to the heat received a may be very much less than r, since most of the heat received and not reflected may pass through without being absorbed, and consequently without affecting the rate of cooling.

Where the body B is not spherical the expression of (28) is not applicable except in the case of a complete inclosure, since then we do not have the relation of $s=4S$ between the surface and the section of the body intercepting the greatest amount of the heat radiated by B', but the relation differs with the rays coming from different parts of the partial inclosure. If, however, B' and B subtend solid angles as seen from each other so small that the rays coming from B' to B may be regarded as parallel, we have from (16)

$$-\frac{dH}{dt}=B(rs\mu^{\theta}-ar'S\psi\mu\theta') \tag{30}$$

in which s and S do not have the relation of the surface of a great sphere to a great circle of it, but one which may differ very much with the different figures of the body.

From (17) we get in the same manner in the case of several bodies radiating heat to B

$$-\frac{dH}{dt}=B(rs\mu^{\theta}-\Sigma_{n}ar'S\varphi\mu\theta') \tag{31}$$

in which all the quantities under the sign Σ_n may be different, except S where the body B is spherical. In case B is spherical and the quantities a, r', and θ' are the same, this case is included in the general expression of (28), the value of ψ then in 29 depending upon the sum of the solid angles subtended by the several bodies.

Effect of gases on the law and rate of cooling.

79. The rate of cooling of a body in a complete inclosure containing a gas is of course more rapid than in a vacuum, since in the latter case the body loses heat by both radiation and the conduction and convection of the gas. The body, being warmer than the inclosure and the gas, gives rise to convective currents by heating the part of the gas in contact with and near its surface. Dulong and Petit made numerous experiments to determine the rate of cooling due to different gases under different tensions and with different temperatures. They found that the rate was not changed by a change of temperature and of density, provided the changes were such as to leave the elasticity the same. The rate therefore can be expressed in a function of the tension and of the difference of temperature between the body and inclosure without regard to its density. Putting V=the rate of cooling due to any gas, they obtained from experiment the following empiric formula:

$$V=np^{c}t^{1.233} \tag{32}$$

in which p is the tension and t is the difference of temperature between the body and inclosure, n a constant which changes with the nature of the gas and the dimensions of the body, and c a constant which is the same for all bodies, but which changes from one gas to another. The exponent 1.233 was found to be the same for all gases. The values of c are: for air, 0.45; for hydrogen, 0.38; for carbonic acid, 0.517; and for olefiant gas, 0.501.

Still more experiments are needed to show whether (32) holds for very low tensions.

The part of the rate of cooling due to air of ordinary pressure in the inclosure of 30^{cm} diameter was a little less than the part due to radiation of a naked thermometer. Different gases were found to have very different effects. Putting the effect due to air equal unity, for hydrogen of the same tension it is 3.45; for carbonic acid, 0.965.

In the case of a silvered thermometer the air was cooled 5 or 6 times as much by the effect of the air as by radiation.

The preceding law (32) was subsequently verified by MM. de la Provostaye and Desains[58] in the case of air in a spherical inclosure, but from experiments made with an inclosure of 24^{cm} and 15^{cm} in diameter they found that the rate of cooling due to the air was greater in the smaller than in the larger inclosure, whatever the state of the surface of the cooling body, and the differences were the greater the more feeble the tensions. The value of the constant n, therefore, seems to depend upon the dimension of the inclosure as well as upon that of the body and upon the nature of the gas.

In a cylinder 6^{cm} in diameter and 20^{cm} in length the rate of cooling due to the air was greater than in the large inclosure from a tension of 760^{mm} down to about 45^{mm}, but the contrary for smaller tensions. In the case of the cylinder they found that the law of (32) did not hold accurately.

For the complete rate of cooling, therefore, in an inclosure containing gas, V, (32), must be added to (22), and CV to (28) when the latter is applied in case of a complete inclosure. The value of n must be determined from experiment in each special case.

Where the body B is placed in the open air in motion with a velocity, v, we can assume, at least for small ranges of temperature, that the rate of losing heat is proportional to the velocity of the current and the difference between the temperature of the body θ and that of the air denoted by θ'. We shall therefore have in this case instead of (32)

$$(33) \qquad V=\frac{kv(\theta-\theta')}{C}$$

in which k is a constant independent of velocity and $(\theta-\theta')$.

Effect of diathermic envelopes.

80. Where the body B is surrounded by a diathermanous envelope, solid, liquid, or gaseous, the rate with which the body loses heat is less than it is where the body has no such envelope. For the heat in pass-

ing out through the envelope is partly reflected back in the numerous irregular reflections and re-reflections, especially if the envelope is gaseous, such as the atmosphere, and a part also is absorbed, one half of which is radiated back to the body and only the other half escapes out through the envelope.

The intensity of radiation from each element ds of the body B with an angle of emission i, if the envelope touches the body, as the atmosphere does the earth, is diminished in the ratio of 1 to p^{ε}, in which ε is the secant of i, just as the intensity of the solar rays is diminished in passing through the atmosphere into the earth's surface, as expressed in (94), § 46, in which A is the intensity before the ray enters the atmosphere. The value of p in any case depends of course upon the thickness of the envelope and its transmissive power, as is seen from (96) of § 47. The relative radiation for all angles of emission i, which in case of no envelope is expressed by (7), now becomes rm, in which

$$m=\frac{\int p^{\varepsilon} \cos i d\sigma}{\pi} \tag{34}$$

omitting here, for the sake of simplicity, the small effect of $\varphi(i)$, which expresses the variation from Lambert's law of cosines, and which vanishes in the case of a lampblack surface and is very small in most cases.

In like manner the rays coming from each element ds' of the inclosure, with an angle of incidence i to any element ds of the body B, are likewise weakened in the ratio of 1 to p'^{ε}, so that instead of the expressions of (21) for the relative absorption for the rays falling upon ds with all angles of incidence in case of no envelope, we have now an, in which

$$n=\frac{\int p'^{\varepsilon} \cos i d\sigma}{\pi} \tag{35}$$

The less the thickness of the envelope and the greater the transmissive power the more nearly p and p' approximate to unity, and consequently when the envelope vanishes m and n become unity. These, therefore, express the ratio between radiation or absorption as affected and not affected by the envelope. If the transmissive powers, which are the complements of α in (96), § 43, are the same for the heat radiated by B, and for that received from the inclosure, then we have $p' = p$, and consequently $n = m$. If we substitute for r and a in the expressions of (28) and (30) rm and an, the factors m and n express the effect of the diathermanous envelope.

Effect of the atmosphere

81. The rate with which the body B loses heat by radiation is expressed by the term $BS\,r\mu^{\theta}$ in (28) and (30). But if the body is not in a vacuum, but in air or some kind of gas, the rate of cooling is increased very much by the conduction, and especially the convective currents which arise while the body B is warmer than the inclosure and the

medium. The effect of this is expressed by V in (32), and consequently the effect on the rate of losing heat is

$$CV = Cnp^{c}t^{1.233}$$

With this added to the second number of (28) we get, by means of (5)

$$(36)\quad -\frac{dH}{dt} = Bs(r\mu^{\theta} + kt^{1.233} - ar'e\mu^{\theta'}) = Bs\mu^{\theta'}(r\mu^{\theta-\theta'} - ar'e + kt^{1.233})$$

in which

$$k = \frac{Cnp^{c}}{Bs\mu^{\theta'}}$$

is a constant independent of t, but depending upon the various other conditions contained in its expression.

By development we get, giving to μ its value, 1.0077,

$$(37)\qquad \mu^{\theta-\theta'} = 1 + .0077(\theta-\theta') + \frac{.0077^{2}}{2}(\theta-\theta')(\theta-\theta'-1) + 01$$

Supposing the temperature of the gas to be the same as the inclosure, we have $t=\theta-\theta'$, and we can put

$$(38)\qquad kt^{1.233} = kt \times t^{.233} = c(\theta-\theta')$$

in which

$$(39)\qquad c = kt^{.233} = \frac{Cnp^{c}t^{.233}}{Bs\mu^{\theta'}}$$

is very nearly a constant, at least for small ranges of t, and may be so regarded where we do not wish to obtain exact quantitative results, but merely to show the general effect of the gas on the rate of cooling. For instance, if $t=10°$, we have $t^{.233}=1.7$, and when $t=20°$, we have $t^{.233}=2$.

By means of (37), using only the first two terms of the expression, and (38), we now get from (36)

$$(40)\qquad -\frac{dH}{dt} = Bs\mu^{\theta'}[(.0077r + c)(\theta-\theta') + r - ar'e]$$

In a complete inclosure we have $e=1$, $r'=1$, and putting $a=r$, as we must in this case, we get

$$(41)\qquad -\frac{dH}{dt} = Bs\mu^{\theta'}(.0077r + c)(\theta-\theta')$$

In the case of no gas in the inclosure, c vanishes. From the experiments of Dulong and Petit, with an inclosure 30cm in diameter, containing air of ordinary tension, the part depending upon c was nearly equal to the part depending upon radiation, and hence in that case $c=.0077r$ nearly. When the radiating power r of the body is small, c is much greater than $.0077r$, and hence in the case of silvered thermometers Dulong and Petit found the effect of the air to be 5 or 6 times greater than that of radiation.

The expression of (40) is applicable to the cooling of a body in the open atmosphere without motion, except that arising from the convective currents due to the higher temperature of the body than that of the air. The value of c, however, would be different, and most probably some less, since according to the observations of MM. de la Provostaye and Desains, the effect of the gas was less in larger inclosures. Where, however, the air is in motion, this convective effect is, perhaps, entirely lost in the far greater one due to the currents, in which case the effect on cooling is expressed by V of (33), and on the rate of loss of heat by $CV=kv(\theta-\theta')$. Using this value of CV instead of the preceding one, and proceeding in the same manner as in the preceding case, we get, instead of (40)

$$(42) \qquad -\frac{dH}{dt}=Bs\mu^{\theta'}[(.0077r+kv)(\theta-\theta')+r-ar'e]$$

In this case the part depending upon kv may be very much greater than that depending upon radiation, according to the velocity of the wind v. In fact it may be so great that the other sensibly vanishes in comparison. This expression is only applicable where the air or gas is of the temperature θ' of the partial inclosure, as in the case of the atmosphere and earth's surface.

Effect of the sun's heat.

82. The effect of the sun's heat may be expressed by the last term of (30), in which case r', θ', and ψ are respectively the radiating power, the temperature, and the solid angle of the sun's disk. But neither r' nor θ' are known, neither can the law of Dulong and Petit, expressed by $\mu^{\theta'}$, be extended to so high a temperature as that of the sun. The intensity of the radiation of the body B' in (30) is $Br'\psi\mu^{\theta'}$. This, in the case of the sun, becomes A, called the solar constant. Hence we have

$$(43) \qquad A=Br'\psi\mu^{\theta'}$$

But the intensity of solar radiation within the earth's atmosphere, we have seen, is $I=Ap^{\epsilon}$ (100), neglecting the very small term arising from the want of homogeneity in the solar rays. In this case, therefore, instead of $Br'\psi\mu^{\theta'}$ in (30) we must put Ap^{ϵ}, and with this substitution the last term of (30) becomes $-aSAp$ for the effect of the solar rays. With this added to the general expression of (28) we get, by means of (5), in the case in which the effect of the sun's rays is added to that of a complete or partial envelope,

$$(44) \qquad -C\frac{d\theta}{dt}=SB\mu^{\theta'}(r\mu^{\theta-\theta'}-ar'e)-a'SAp^{\epsilon}$$

in which the absorbing power of the body for solar heat is distinguished by an accent, since some bodies, § 70, absorb more of solar heat than of the heat from an inclosure or any body nearly of their own temperature.

Except for a very short time in which the direction of the sun's rays does not change much with regard to the body B, this expression is applicable to a spherical body only, since with any other figure S, which is the area of the section of the body intercepting the most rays, does not remain constant. In the case of a spherical body we have $s=4S$, and with this relation the expression can be simplified in this case.

In this expression S, as defined in § 72, is the section of the body B which intercepts the greatest amount of the solar rays, and which, in the case of a spherical body, becomes the area of a great circle.

At the top of the atmosphere, or in case of no atmosphere, p^{ϵ} becomes unity.

Where the cooling body is not *in vacuo*, and exposed to the sun's rays, we must subtract the term $a' S A p^{\epsilon}$ from the last member of (40), and we then get

$$(45) \qquad -C\frac{d\theta}{dt}=sB\mu^{\theta'}\,[(.0077\,r+c)\,(\theta-\theta')+r-ar'e]-a'SAp^{\epsilon}$$

In case the body is exposed in currents in the open air to the sun's rays, we must subtract this term expressing the effect of the sun's rays from the second member of (42) and we thus get

$$(46) \qquad -C\frac{d\theta}{dt}=sB\mu^{\theta'}\,[(.0077\,r+kv)\,(\theta-\theta')+r-ar'e]+a'SAp^{\epsilon}$$

In both these expressions with a complete inclosure, as has been explained, we have $r-ar'e=0$, but in the last we cannot have a complete inclosure unless the atmosphere is too cloudy to permit radiation from the earth's surface through it.

In these expressions e, (29), may include either a whole or a partial inclosure, but if the former, the inclosure must be diathermanous, as in the case of a thermometer *in vacuo* with a glass inclosure, else the sun's rays would be excluded. If the inclosure is not perfectly diathermanous the heat cut off by it must not be taken into account. In the case of bodies near the earth's surface, in open space, the earth is a partial solid inclosure subtending a solid angle of a hemisphere, in which case $\psi=2\pi$ in (29), but on the other side the atmosphere, when the body is within it, forms also an inclosure imperfect, when clear, radiating and reflecting back to the earth nearly or quite as much heat as would be radiated by a solid envelope of the same temperature as the earth's surface. This, of course, differs with different states of the atmosphere, but when it is clear the rate of cooling is greater than it would be in case of a complete inclosure or of very cloudy weather, the term $a'Ap^{\epsilon}$ remaining the same, or being entirely absent, as at night.

III.—TEMPERATURE OF BODIES NEAR THE EARTH'S SURFACE.

General expressions of the temperature and its variations.

83. The conditions which must be satisfied in determining the temperature of bodies exposed near the earth's surface to the sun's rays are expressed by the equations (44), (45), and (46). The first is applicable where the body is *in vacuo;* the second, where the body is in air or gas of some kind in an inclosure without motion except convective currents arising from the heat of the body; and the last, where the body is exposed in currents of the atmosphere. The first is included in either of the last two, so far as their approximate applications extend, which is only to quantities of a second order in the developments upon which they depend. The first is retained, therefore, for a more accurate application in special cases. By putting $c=0$ in (45), or $kv=0$ in (46) we get (44), except the small terms of the third and lower orders omitted in the developments in obtaining (45) and (46).

Having given the surroundings of a body, consisting of an inclosure either complete or partial according to the value of e; its temperature θ' and radiating power r'; the intensity of solar radiation Ap^{ε}; the magnitude, shape, radiating and absorbing power and capacity for heat, of the body; all of which enter into the equations of conditions, what is now required is to determine the temperature θ of the body under the circumstances.

The normal intensity of solar radiation Ap is a variable function of the time t, since it contains A, which varies slightly with the sun's varying distance from the earth; p, which varies abnormally with different states of the atmosphere; and $\varepsilon=\sec z$, which depends upon the sun's zenith distance at different times. The vertical intensity upon any part of the earth's surface is $Ap^{\varepsilon} \sin z$, which may be represented by expressions of the form of (21) and (22). We can also suppose Ap^{ε} to be represented by a similar expression, with regard to its variations of both diurnal and annual periods and their submultiples, and can, therefore, put

$$(47) \qquad Ap^{\varepsilon}=\Sigma K_s \cos\,(u_s t-k_s)$$

in which u_s is the rate of change of the angle for any inequality indicated by the suffix s, and K_s and k_s are constants generally unknown, or at most only approximately.

With this expression (44) becomes

$$(48) \qquad -C\frac{d\theta}{dt}=sB(r\mu^{\theta}-ar'e\mu^{\theta'})-a'S\Sigma K_s\cos(u_s t-k_s)$$

and (46) becomes

$$(49) \quad -C\frac{d\theta}{dt}=sB\mu^{\theta'}[(.0077r+kv)(\theta-\theta')+r-ar'e]-a'S\Sigma K_s\cos\,(u_s t-k_s)$$

in which the quantities under the sign Σ, varying in the different components of which the characteristic is s, are as in (47).

Equation (45) becomes the same as this with c instead of kv.

For reasons already given, § 82, these expressions are applicable in general in the case only in which the body B is spherical.

The solution of (48) or (49) is necessary for determining the temperature θ of the body under the conditions expressed by the equation.

If $\mu^{\theta}=\mu^{\theta'} . \mu^{\theta-\theta'}$ in (48) is developed, as in (37), and terms of the third and lower orders neglected, it becomes the same as (49). In an approximate solution, therefore, in which these terms are neglected, which is all that can be attempted here, the former becomes simply a special case of the latter in which we put $kv=0$, as in the case of a vacuum. The solution of the latter, therefore, will be applicable also to the former in all cases in which the solution cannot be completely made.

The solution of (49) must necessarily give an expression of θ of the form of (47), and we can therefore put

$$\theta=\theta_0+\Sigma A_s \cos (u_s t-\varepsilon'_s) \tag{50}$$

Where the earth's surface and the atmosphere become the inclosure with temperature θ', this likewise becomes a variable, since it is changed for the same reason as the temperature of the body. We can in this case put

$$\theta'=\theta'_0+\Sigma A'_s \cos (u_s t-\varepsilon'_s) \tag{51}$$

in which ε' will in general differ but little from ε, since the maximum of θ will in general occur nearly at the same time as that of θ'.

We can put in (49), neglecting terms of the third and lower orders,

$$\mu^{\theta'}=\mu^{\theta_0'}. \mu^{\theta'-\theta_0'}=\mu^{0_0'}\Big[1+.0077 \ (\theta'-\theta_0')\Big] \tag{52}$$

With this in (49), we get by means of (50) and (51),

$$\begin{aligned} -C\frac{d\theta}{dt}=\ & sB\mu^{\theta_0'} (.0077r+kv) \ \Sigma A_s \cos (u_s t-\varepsilon_s) \\ & +sB\mu^{\theta_0'} \ [(.0077r+kv) \ (\theta_0-\theta'_0)+r-ar'e] \\ & -a'SK_0-sB\mu^{\theta_0'} \ (.0077 \ ar'e+kv) \ \Sigma A'_s \cos (u_s t-\varepsilon'_s) \\ & -a'S\Sigma K_s \cos (u_s t-k_s) \end{aligned} \tag{53}$$

By combining the last two terms by the usual method,* the resultant may be put into the form:

---

* The general form of such reduction is given in (35) and (36) § 276, from which we have, in the case of two terms only

$$A_1 \cos \phi_1+A_2 \cos \phi_2=R \cos (\phi_1+\beta)$$

in which

$$R=\sqrt{A_1+A_2 \cos (\phi_2-\phi_1)+A_2 \sin (\phi_2-\phi_1)} \qquad \tan \beta=\frac{A_2 \sin (\phi_2-\phi_1)}{A_1+A_2 \cos (\phi_2-\phi_1)}$$

Putting in the last two terms of (53) for each pair of terms of which the characteristic is s, $A_1=a'K_s$, $\phi_1=u_s- k_sA_2=4B\mu^{\theta_0}$ $(.0077 \ ar'e+kv)SA'$ and $\phi_2=u_st-\varepsilon_s$, substituting these values of A_1, A_2, ϕ_1 and ϕ_2, and putting K' for R and $k_s-k'_s$ for β, we get the expressions of (55). Hence $-k'_s=-k_s+\beta$ and $st \ k'_s$ is the epoch of the resultant, which differs from k_s by the effect of β.

$$(54)\qquad S\Sigma K'_s\cos(u_st-k'_s)=S\Sigma K'_s\cos u_st\cos k'_s+S\Sigma K'_s\sin u_st\sin k'_s,$$

in which, putting $s=4S$, as we must in case of a sphere,

$$(55)\qquad K'_s=\sqrt{\begin{aligned}&[a'K_s+(.0077\,ar'e+kv)\,4B\mu^{\theta'_0}A'_s\cos(k_s-\varepsilon'_s)]^2\\&\qquad+[(.0077\,ar'e+kv)\,4B\mu^{\theta'_0}A'_s\sin(k_s-\varepsilon'_s)]^2\end{aligned}}$$

$$\tan(k_s-k'_s=\frac{(.0077\,ar'e+kv)4B\mu^{\theta'_0}A'_s\sin(k_s-\varepsilon'_s)}{a'K_s+(.0077\,ar'e+kv)4B\mu^{\theta'_0}A'_s\cos(k_s-\varepsilon'_s)}$$

From (50) we get,

$$(56)\qquad \begin{aligned}\theta&=\Sigma A_s\cos[(u_st-k'_s)-(\varepsilon_s-k'_s)]\\&=\Sigma\alpha_s\cos(u_st-k'_s)+\Sigma\beta_s\sin(u_st-k'_s)\end{aligned}$$

in which

$$(57)\qquad \alpha_s=A_s\cos(\varepsilon_s-k'_s)\qquad\qquad \beta_s=\Sigma A_s\sin(\varepsilon_s-k'_s)$$

From this we also get

$$(58)\qquad -\frac{d\theta}{dt}=\Sigma u_s\alpha_s\sin(u_st-k'_s)+\Sigma u_s\beta_s\cos(u_st-k'_s)$$

Substituting the second form of (54) for the last two terms of (53), this latter becomes by means of (56) and (58)

$$(59)\qquad \begin{aligned}0=sB\mu^{\theta'_0}\{&[.0077r+kv][\Sigma\alpha_s\cos(u_st-k'_s)-\Sigma\beta_s\sin(u_st-k'_s)]\\&+S\beta\mu^{\theta'}[(.0077r+kv)(\theta_0-\theta'_0)+r-ar'e]\}\\&-C\Sigma u_s\alpha_s\sin(u_st-k'_s)+Cu_s\beta_s\cos(u_st-k'_s)\\&-S\Sigma K'_s\cos(u_st-k'_s)-S\Sigma K'_s\sin(u_st-k'_s)-a'SK_0\end{aligned}$$

For the constant terms under the sign Σ, depending upon the constant in (47), we must put $u_s=0$ and $k'_s=0$.

By equating the coefficients of the cosines and the sines, and likewise the constant terms independent of the time, we get the following three equations:

$$(60)\qquad \begin{aligned}&0=4B\mu^{\theta'_0}[(.0077r+kv)(\theta_0-\theta'_0)+r-ar'e]-a'K_0\\&0=sB\mu^{\theta'_0}(.0077r+kv)\alpha_s-Cu_s\beta_s-SK'\\&0=sB\mu^{\theta'_0}(.0077r+kv)\beta_s-Cu_s\alpha_s\end{aligned}$$

84. From the first of these equations we get for the difference between the constant parts of θ and θ' in (50) and (51), since $4s=S$ and $r=a$

$$(61)\qquad \theta_0-\theta'_0=\frac{a'SK_0}{(.0077r+kv)sB\mu^{\theta'_0}}-\frac{r(1-r'e)}{.0077r+kv}$$

The first term expresses the part of $(\theta_0-\theta'_0)$ depending upon the constant K_0 in (47), and the last one the part independent of solar radiation but arising from the incompleteness of the inclosure on account of which the body stands at a lower temperature than that of the partial inclosure. In a complete inclosure e, by (29), becomes unity and, § 74,

$r'=1$. Where the body and the inclosure have the same temperature, or nearly, we also have sensibly $r=u$. In this case, where $K_0=0$—that is, where there is no heat received from the sun—we have $\theta_0-\theta'_0=0$. The body then stands at the same temperature as the inclosure.

In the preceding expressions θ' being assumed as the temperature of either the air or the earth's surface, the incompleteness of inclosure represented by e must be understood not only deficiency of radiating matter surrounding the body receiving radiations from the earth's surface and the atmosphere, but likewise deficiency in the radiations arising from an average temperature of the atmosphere and the earth's surface lower than the temperature of θ'.

If we put θ''=the value of θ, where K_0 is equal 0, we get from (61)

$$\theta''=\theta'_0-\frac{r(1-r'c)}{.0077r+kv} \tag{62}$$

in which θ'' may be called *the shade temperature* since it is the static temperature of the body independent of any solar radiation, but depending only on the radiations of the inclosure, complete or partial, and becomes the same as that of the inclosure where this is complete.

From this, and 61, we get

$$\theta_0-\theta''=\frac{a'SK_0}{(.0077r+kv)sB\mu^{\theta'_0}}. \tag{63}$$

From the last two of equations (60) the values of α and β may be determined, and then from these the values of A and $(\varepsilon-k)$ in (57), from which, with the known value of k', the value of ε becomes known. The values of A and ε being known, we obtain from (50) the value of θ, the temperature of the body, at any given time t. Putting for brevity

$$p=\frac{(.0077r+kv)sB\mu^{\theta'_0}}{Cu_s} \qquad q=\frac{a'SK'}{Cu_s}$$

the last two of (60) give

$$\alpha_s=p\beta_s \qquad \beta_s=q-p\alpha_s$$

The solution of these gives

$$\alpha_s=\frac{pq}{1+p^2} \quad , \quad \beta_s=\frac{q}{1+p^2}$$

Substituting for α_s and β_s their values in (57), and for p and q the expressions above, we then obtain from them the following expressions:

$$(64) \quad \begin{cases} A_s=\dfrac{SK'_s}{\sqrt{[(.0077r+kv)sB\mu^{\theta'_0}]^2+(Cu_s)^2}} \\ \tan(\varepsilon_s-k'_s)=\dfrac{Cu_s}{(.0077r+kv)sB\mu^{\theta'_0}} \end{cases}$$

Putting

R=the time, called *the retard*, by which the maximum of any inequality in the temperature, of the form of (50), follows the time of the maximum phase of the corresponding inequality of radiation from the sun and the inclosure, (54),

we get from (47) and (54),

$$R=\frac{\varepsilon_s-k'_s}{u_s} \tag{65}$$

When ε_s-k_s is so small that we can use the arc instead of the tangent, we have from (64) and (65),

$$R=\frac{C}{(.0077r+kv)sB\mu^{\theta'_0}} \tag{66}$$

The preceding expressions are applicable to each of the inequalities of the intensity of solar radiation in (47), where we confine ourselves to an approximate solution only by neglecting the terms of the third and lower orders in the developments of (37) and (52); but if we attempt a second approximation by taking in another order of terms, the problem, and likewise the resulting expressions of A and tan $(\varepsilon-k')$ in (64), become very complex. It is seen from these developments that if we confine ourselves to small ranges of $(\theta-\theta')$ the neglected terms are small. For ordinary ranges of temperature oscillations on the earth's surface the effects of these neglected terms are small, but for large ranges the expressions become inaccurate.

In case of no temperature inequalities of the inclosure A'_s and ε'_s in (55) vanish and we have $K'_s=a'K_s$ and $k'=0$. We thus have, instead of (64),

$$\left.\begin{aligned} A_s&=\frac{a'SK_s}{\sqrt{((.0077\,r+kv)\,sB\mu^{\theta'_0})^2+(Cu_s)^2}} \\ \tan\varepsilon_s&=\frac{Cu_s}{(.0077\,r+kv)\,sB\mu^{\theta'_0}} \end{aligned}\right\} \tag{67}$$

If the oscillations of temperature of the inclosure coincide in epoch with those of the intensity of solar radiation we have from (47) and (51) the angle $k_s=\varepsilon_s$. With this relation we get from (55),

$$A'_s=\frac{K'_s-a'K_s}{(.0077\,ar'e+kv)\,4B\mu^{\theta'_0}}$$

If in (64) we put the usually very small term $Cu_s=0$, we get, since then $\varepsilon'_s-k_s=0$, and consequently by (55) $K'_s=a'K_s$,

$$A_s=\frac{a'K_s}{(.0077\,r+kv)\,4B\mu^{\theta'_0}}$$

In a complete inclosure we have $r'=1$ and $e=1$. Hence in this case we get, by subtracting the former of the last two equations from the latter,

$$(68) \qquad A_s - A'_s = \frac{a'K_s}{(.0077r+kv)4B\mu^{\theta_0'}}$$

This expression of $A_s-A'_s$ is in general very nearly the same as that of A_s in (67), on account of the usually very small value of Cu_s in the latter. The epochs of the temperature oscillations of the air and earth's surface are very nearly the same as those of the body, so that, unless e should differ very much from unity, or Cu_s should be very large, (67) may be used instead of (64), if it is understood that A_s and ϵ_s are the amplitude and epoch of the temperature of the body relative to that of the atmosphere and earth's surface. And this is especially the case upon the ocean, where the amplitudes of the temperature oscillations of the air and earth's surface, both diurnal and annual, are very small; but even in the interior of the continents, where these oscillations may be quite large, the error would in general be very small. In what follows, therefore, we shall use (67) instead of the more complex expressions of (64), and understand the temperature, θ, of the body to be that relative to the temperature of the air and earth's surface, where the latter is not constant.

Deductions from the preceding expressions.

85. From (61) and (67) we readily see the effect of the various circumstances which enter into the expressions, upon the constant of temperature $(\theta_0-\theta_0')$, and the amplitudes and epochs of the temperature oscillations of the body, corresponding to any given inequalities in intensity of solar radiation in (47). First, let us consider, where the body is in the atmosphere near the earth's surface, the effects of ventilation. If in (61) and (67) we make kv infinitely great we get $\theta-\theta'_0$, A_s, and ϵ_s, each equal to 0, and hence the temperature of the body is reduced to that of the temperature of the air. But a finite, and perhaps even a small value of v, may often so reduce the temperature of the body as to make it *sensibly* the same as that of the air. It is seen, where kv is very great, that the value of v required to reduce the difference to an insensible quantity is proportional to the intensity of solar radiation represented by K_0 and K_s and the absorbing power a' of the body for solar heat; for when kv becomes very great the term in the denominators containing r as a factor can be neglected without much error. The greater, therefore, the intensity of solar radiation and the absorbing power of the body for solar heat, the greater must be the ventilation and very nearly in proportion. In obtaining the air temperature, therefore, by means of the sling thermometer, we see the great advantage of doing this in the shade rather than in the sunshine, since in the former case K_0 and K nearly vanishes, and merely represents the part of the solar

rays reflected by the atmosphere to the thermometer where it is not completely sheltered from them. Also the advantage of having a thermometer with a small value of a', as in the case of a silvered thermometer, since in that case the numerators in (61) and (67) are comparatively very small, and consequently only a proportionate amount of ventilation is required to reduce the temperature of the body sensibly to that of the air.

The value of k in (61) and (67) depends mostly upon the density of the air or other fluid in which the body may be immersed. With water instead of air k would have a very large value and only a very small value then of v would be necessary. In this case we could substitute c in (39) instead of kv, so that the temperature of the body would be sensibly the same as that of the fluid, merely by the effect of conduction and convection.

The values of the expressions of (61) and (67) also depend upon the ratio between S and s, which in the case of a round body is that of 1 to 4. These expressions are applicable in the case of a round body only, as has been stated, unless we understand S to be the average value of the intercepting normal section for all positions of the body with reference to the sun's rays. This in a cylindrical body would be much less in proportion to s than in the case of a round body, and the more so the longer the body in comparison with its cylindrical diameter. The temperature of a cylindrical thermometer is therefore more readily reduced approximately to that of the air than that of a spherical one.

Where kv in the denominators of (64) is very large, the term containing r depending upon the radiation of the body becomes so small in comparison that it may be neglected, and we then have $(\theta_0 - \theta')$ in proportion to the absorbing power of the body. A less value of kv, therefore, is required to reduce $(\theta_0 - \theta')$ to an insensible quantity when the body has a small than when it has a great absorbing power.

The best kind of a sling-thermometer, therefore, as deduced from the preceding equations of the conditions determining temperature, is one of a cylindrical form with a silvered surface, and the best place for slinging it is where the fewest rays of the sun reach it, either directly or by reflection. The absorbing power of a silver surface for heat from a low temperature source is only about 0.03, but for the sun's heat, from what we have seen, §70, it may be as much as 0.08. But even this would be only about one twelfth of that of lamp-black, and consequently only about one-twelfth as much ventilation would be required in the case of the silvered as in the case of the black-bulb thermometer of the same form.

In the case of a body in quiet air v in the preceding expressions vanishes, but there is still an effect arising from convection and conduction which is indicated by using c instead of kv. The value of this constant depends upon the nature of the air or gas, its density, and the size of the inclosure, as has been shown in the experiments of Dulong and Petit and those of MM. Provostaye and Desains, §79. The effect of this in the case of ordinary air is to more than double the rate of cooling which the body would have in a vacuum from radiation alone, and consequently to cause a body of maximum radiating power to stand where in a state of static equilibrium of temperature in the sunshine, at a temperature which differs from that of the inclosure only about half as much as that of a black-bulb thermometer *in vacuo*, as is observed in actinometers of the forms of Secchi's, Soret's, and Violle's, in which the thermometers are exposed in the air and not *in vacuo*.

Nocturnal cooling of bodies.

86. If in (64) we put $K_0=0$, we get, by putting $a=r$, as we must in this case,

$$\theta'_0-\theta_0=\frac{r(1-r'e)}{.0077\,r+kv} \qquad (69)$$

The first member of this equation expresses the difference between the static temperature θ_0 of the body and that of the air θ'_0, where the body is not exposed to solar radiation, as during the night. The satisfying of the conditions of this equation determines the temperature of the body where that of the air θ'_0 and the quantities entering into the second member are known. The difference of temperature $(\theta'_0-\theta_0)$ is said to be due to nocturnal radiation, but it is seen from the second member of (67) that the radiations in the case are simply the ordinary ones which have entered into all the preceding more general expressions. By nocturnal radiation, therefore, we must not understand that anything more is meant than ordinary radiation producing its effect under nocturnal conditions, as expressed in (69).

The effect is due to imperfection of inclosure, as defined in § 84. If in (69) we put $e=1$, as we must by (29), where the inclosure is perfect, in which case we have seen, § 74, we also have $r'=1$, then the second member of (69) vanishes, and we have $\theta'_0-\theta_0=0$—that is, we have none of the effect attributed to nocturnal radiation. When, however, the body is exposed in the air near the earth's surface, subtending in that case a solid angle of half a hemisphere, this surface forms the inclosure on the one side, but the atmosphere around and above the body, especially when very clear, does not complete the inclosure, and the effect is the same as if some part of the inclosure were entirely wanting. The more incomplete the inclosure—that is, the more the value of e differs from unity—the greater, by (69), is the value of $(\theta'_0-\theta_0)$.

Where there is a static equilibrium of temperature a complete inclosure returns to the body by radiation and reflection as much heat as it receives from it, but if a part is transmitted into space, the body receives just so much less than it radiates, and the inclosure is incomplete in this ratio. The radiation of the body *in vacuo* in any direction with reference to the zenith through a unit of solid angle being regarded as unity, the amount transmitted through the element of solid angle $d\sigma$ into space is $p^\epsilon d\sigma$, in which p has the value in (34) depending upon the transmissive power of the atmosphere for radiations from terrestrial bodies. We therefore have for the whole amount of heat transmitted into space $\int p^\epsilon d\sigma$, in which the integration must be extended over a hemisphere with the zenith in the center, if the earth's surface is a plane, but less if the body should be in a valley between mountain ranges. The whole amount of radiation from the body into space through a vacuum, and consequently the amount of heat received from a perfect inclosure, is $\int d\sigma$, integrated through two hemispheres, which gives 4π. We therefore have

$$e=\frac{4\pi-\int p^\epsilon d\sigma}{4\pi}=1-\frac{1}{4\pi}\int p^\epsilon d\sigma=1-m \tag{70}$$

in which

$$m=\frac{1}{4\pi}\int p^\epsilon d\sigma \tag{71}$$

Where there is no transmission into space we have $p=0$, and consequently $e=1$, which by (29) is its value for a perfect inclosure. The value of p for solar heat in a clear atmosphere is about 0.75, but for terrestrial heat, though unknown, it is supposed to be very much less. If its value were known the last term in (70) could be obtained by some means of integration.

Pouillet's actinometer.

87. It may, however, be approximately determined from certain experiments made by Pouillet[56] with an instrument which he called an *actinometer*. It consisted of several layers of swan's-down resting one upon another, so as not to experience any compression, the swan's skin itself forming the foundation of the layers. This system was inclosed in a first cylinder of silver plate enveloped also in swan's skin, and contained in a larger cylinder. A thermometer was placed on the upper down in such a position with reference to the sides of the top part of the cylinder that it could radiate toward two-thirds of the hemisphere of the sky. Having observed the temperature of the thermometer on the swan's-down, he instituted an artificial sky of maximum radiating power, with such temperature that the heat radiations would keep the temperature of the thermometer the same as when it received, instead, the radiations from the clear sky. This he called the *zenithal temperature*. If this were the same as the temperature of the air and the thermometer suspended in it at the same height, the thermometer on the swan's-down would receive as much heat from the artificial sky as it radiated to it, § 86. With the zenithal temperature it radiates to the thermometer as much as it receives and does not transmit into space, supposing it had the diathermancy of a clear atmosphere. Hence the

difference between the radiations of this artificial sky with the temperature of the air, and with zenithal temperature, is the part of the radiation from the thermometer which passes into space. Putting θ_z for the zenithal temperature, these radiations are as μ^{θ} to μ^{θ_z}, and therefore we shall have by integrating through two-thirds of a hemisphere

$$\frac{\int p^{\epsilon} d\sigma}{\int d\sigma}=\frac{\int p^{\epsilon} d\sigma}{\frac{4}{3}\pi}=\frac{\mu^{\theta}-\mu^{\theta_z}}{\mu^{\theta}}=1-\mu^{\theta-\theta_z}=.0077\ (\theta-\theta_z)$$

Hence we get, (71),

(72) $$m=\frac{1}{4}\pi\int p^{\epsilon} d\sigma=.00257\ (\theta-\theta_z)$$

where the integration is extended through two-thirds of a hemisphere, as required in Pouillet's experiments.

The average of the values of $(\theta-\theta_z)$ from 35 observations made by Pouillet[56] at different hours, evening and morning, of fine clear nights, is 16° C. With this value (72) gives

(73) $$m=0.041$$

Where bodies are exposed on a plane in the open air, the value of m is somewhat greater, but as the remaining third of the hemisphere lies around near the horizon where ϵ is large, the integral, on account of the smallness of p for terrestrial radiations, is not very much increased by extending it through this remaining third. In this case we may perhaps put, with small error,

(74) $$m=0.05$$

With this value we get from (70), for clear weather, $e=0.95$, indicating almost a complete inclosure. With a complete inclosure we also have $r=1$, and with an incomplete one it must decrease somewhat in the ratio of e, for small deficiencies of inclosure. We can therefore put approximately $r'e=e^2=1-2m$, (70), and with this (69) becomes

(75) $$\theta'_0-\theta_0=\frac{2rm}{.0077r+kv}$$

in which m must have the value given by (71) with the integration extended through the whole solid angle ς subtended by the open sky, or through a whole hemisphere where the body is exposed in the open air above a level plane.

Effect of cloudiness and increase of altitude upon nocturnal cooling.

88. Where the atmosphere is adiathermanous to terrestrial heat radiations, as in cloudy weather, in which case we have $p=0$, and by (71), $m=0$; or, where the air has motion, such as to give a very large value to kv in the denominator of (75); $(\theta'_0-\theta_0)$ in both cases sensibly vanishes, and it is then said that there is no *nocturnal radiation.*

Where the air about the body is perfectly quiet v vanishes, and instead of kv we must put c, and (73) then becomes

$$(76) \qquad \theta'_0-\theta_0=\frac{2rm}{.0077r+c}$$

For very small values of v neither formula is strictly applicable, but they are generally used in the extreme cases, (75), and all previous expressions in which kv occurs, to show the effect of ventilation, and (76) where there is no motion of the air, to indicate the law and amount of nocturnal cooling.

From an inspection of (76), in connection with (71), it is readily seen that there can be no nocturnal cooling in cloudy weather, and that in clear weather it must be somewhat in proportion to the clearness of the atmosphere, or at least to the proportion of heat, represented by m, which escapes through the atmosphere. As we ascend a high mountain p in (71), as is seen from (96), Chap. 1, increases from a small value with decrease of barometric pressure, and soon becomes comparatively large, and of course the powers of p much more so, so that the value of m increases rapidly with increase of elevation. This is very evident when we consider that at sea-level in clear weather its value is approximately that of (74), but for the top of the atmosphere, where $p=1$, (71) gives $m=0.5$, or ten times greater. But not only does the value of m increase with increase of elevation, but the constant c decreases, somewhat in proportion to the rarity of the atmosphere. The value of $(\theta'_0-\theta_0)$, therefore, is increased from both causes as we ascend to greater altitudes, and hence arises the comparatively large amount of nocturnal cooling of bodies observed on mountain tops.

The value of c in (76) can only be determined by observation. It depends upon the part of the rate of cooling of a body in the atmosphere in comparison with that due to radiation, which, we have seen, § 79, varies with the density of the air and size of inclosure in a complete inclosure, and no doubt varies considerably in incomplete or open-air exposures.

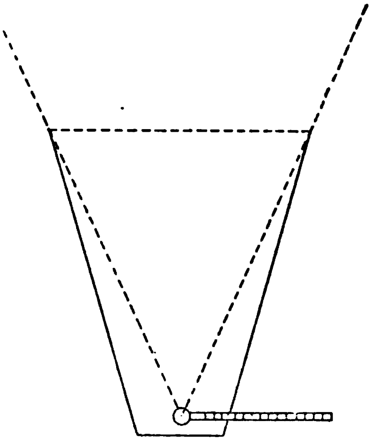
In Pouillet's actinometric experiments he found the following relation between the air temperature θ'_0, that of the thermometer on the swan's-down, θ_0, and the zenith temperature θ_z:

$$(77) \qquad \theta_0'-\theta_z=\tfrac{9}{4}\,(\theta_0'-\theta_0)$$

The average of his determinations of $(\theta_0'-\theta_z)$, we have seen, was 16° C. Hence, in the case of his actinometer, we have in (76) $(\theta_0'-\theta_0)=7°$ very nearly. His thermometer seems to have been an ordinary thermometer, for which we will put $r=0.85$. With these values and the value of m in (73), which is the value for an exposure to two-thirds of the sky hemisphere, (76) gives $c=.0034$. This makes the rate of cooling and the effect in (76) depending upon c only about half that depending upon radiation, which is smaller than in the experiments of MM. de la

Provostaye and Desains, but may not be much in error for Pouillet's actinometer, since it was found to be less for large than for small inclosures. But perhaps the value of *m* used is a little too small.

FIG. 7.



Melloni's experiments.

89. Experiments similar to those of Pouillet with his actinometer were made by Melloni[60] with vessels formed of tin-plate, and of the shape of a truncated cone inverted, as represented in the figure, the radius of the lower ends being 2 centimeters, and that of the upper ends 7 centimeters, and the height 8 centimeters. They were supported by tripods 50 centimeters in height, formed of slender tin-plate tubes, which communicated very little heat from the ground, and answered the purpose of the swan's-down in Pouillet's actinometer, which is a poor conductor.

The thermometers were coated with armatures of different radiating powers, and the thermometers within gave the temperatures to which these armatures were cooled, and the differences between these temperatures and that of the air was taken as the relative radiating powers of the substances forming the different armatures.

The following are some of the results obtained by this method:

Name of radiating body.	Temperatures.		Difference $\theta'_0-\theta_0$	Ratios.	r
	Of the body.	Of the air.			
	°	°	°		
Lamp-black	14. 21	17. 61	3. 40	1. 00	1. 00
Carbonate of lead......	13. 94	17. 30	3. 36	. 99	0. 98
Varnish	14. 10	17. 42	3. 30	. 97	0. 93
Isinglass	13. 67	16. 93	3. 26	. 96	0. 91
Glass	13. 63	16. 79	3. 16	. 93	0. 85
Plumbago	13. 60	16. 52	2. 92	. 86	0. 73
Polished silver ...	17. 383	17. 522	*0. 139	. 04	0. 018

* Instead of this difference Melloni used 0.108, which seems to have arisen from an error in subtraction.

In these experiments the portion of sky toward which the substances radiated, as seen from the preceding figure, was very much less than in the case of Pouillet's actinometer, but being near about the zenith, the value of *m* given by (71) would be in a much greater ratio than the proportion of sky subtending the solid angle of radiation, especially where the value of *p* is as small as in terrestrial radiations. We may therefore suppose its value would be only about one-half, or less, of what it is in the case of Pouillet's actinometer. With the value of m taken at one-half the value in (73) and the value of $r=0.85$ and $c=.0034$ as determined from Pouillet's experiments, we get from (76) for glass $\theta'-\theta_0=3.50^\circ$ instead of 3.16 in the table above.

From (76) we get

$$r=\frac{c\,(\theta_0'-\theta_0)}{2m-.0077\,(\theta_0'-\theta_0)} \tag{78}$$

This expression gives the radiating power of the body from the observed value of $(\theta_0'-\theta_0)$ where the constants c and m are known.

The value of c in cases in which the body is suspended in air, as in the case of the experiments of Dulong and Petit and of MM. de la Provostaye and Desains, and the various kinds of actinometers in which the thermometers are exposed in the air and not *in vacuo*, seems to be such as to make its effect about equal to that of radiation, and hence in the case of a black-bulb thermometer in which $r=1$, as seen from (76), it must be about .0077. In the case, however, of the thermometer in Pouillet's and Melloni's experiments, in which it lies in the former on the swan's-down, and in the latter at the bottom of the cup, there can not be any convective currents, as in the other cases, but the value of c must depend entirely upon the conduction of the air around the thermometer, and therefore should probably be much less, though not so small as the value .0034 which we have obtained from Pouillet's experiments.

Putting $c=.006$, and $r=1$, we get from (78), with the first difference in Melloni's table, 3.42°, the value of $m=.0233$. Using now this value of m and the preceding assumed value of c, we get from (78), with the observed values of $(\theta_0'-\theta_0)$ in the preceding table, the corresponding values of r in the last column of that table.

These values, it is seen, differ considerably from those of Melloni's, but agree better with those obtained by Leslie and others. It is true there is considerable uncertainty with regard to the proper values of the constants c and m to be used, but this uncertainty is not so great as to affect the results very much, though they must be regarded as being only approximate. From a mere inspection, however, of (78), it is seen that we cannot assume, as Melloni has done, that observed values of $(\theta_0'-\theta_0)$ are relative measures of radiation, unless the value of the constant m in the denominator were so great as to make the term containing $(\theta_0'-\theta_0)$ as a factor, infinitely small in comparison with the other. In fact there is no admissible value of c, and corresponding value of m obtained upon the conditions that $r=1$ for a lamp black surface, which would give values of r differing greatly from those we have obtained with the assumed value of $c=.006$.

If the cups used by Melloni had been wider at the top, so that the solid angle indicated by the dotted lines in the figure had included more of the sky, as in the case of Pouillet's actinometer, the observed effect would have been much greater, for this, by (71), would have increased the value of the constant m in (76). With the solid angle in the case of the actinometer, including two-thirds of the sky hemisphere, the effects were more than double, and would have been still greater if Pouillet had had the true air temperature. This was obtained by means of an ordinary thermometer suspended in the air at the height of the actinometer. But this thermometer, just as the one on the swan's-down, and for the same reason, indicated a temperature lower than that of the atmosphere, but the difference was not nearly so great.

We have seen that in both the actinometer and Melloni's cups the atmosphere above and around so nearly completes the inclosure that the radiating power of the partial inclosure, the swan's-down and sides of the upper part of the inclosing cylinder in the one case, and the cups in the other, scarcely comes into account, since the less the radiated the more the reflected heat, the two together being nearly the same as would be radiated by a lamp-black surface, as they are quite in a complete inclosure, § 74. The principal advantage of the swan's-down in the cylinder, and of the cups, is to prevent convective currents. In the case of the thermometer suspended in the air, as fast as the air immediately around the bulb becomes cooled down below the rest, it drops down and other warmer air takes its place, so that a current is established which continually carries away the cold air and conveys warmer air to it. In the actinometer and in the bottom of the cups, the cold air settles to the bottom and remains entirely stagnant, and the thermometer bulb cools down to a temperature at which heat is *conducted* from the inclosure to the bulb as fast as it is radiated by the bulb through the atmosphere into space, the effect of convective currents not coming into account.

Temperatures in case of no atmosphere.

90. The results deducible in this case from the general expressions of (61) and (67) are not only very surprising but also instructive with regard to the great effect of the atmosphere upon the temperatures of bodies near the earth's surface. Putting, as we must in this case, $kv=0$, we get from (61), where θ' is constant and equal θ'_0, by putting $s=4S$ and giving B its numerical value 1.146,

$$(79) \qquad \theta_0-\theta'=\frac{a'K_0}{.0353r\mu^{\theta'}}-\frac{1-r'e}{.0077}$$

and from (67)

$$(80) \qquad A_s=\frac{a'SK_s}{\sqrt{(.0077rsB\mu^{\theta'})^2+(Cu_s)^2}} \qquad \tan\epsilon_s=\frac{Cu_s}{.0077rsB\mu^{\theta'}}$$

But since (79) depends upon the developed expression of $\mu^{\theta-\theta'}$ in (37), in which terms of the third and lower orders of terms are neglected, it is only approximate and becomes inaccurate when the range of $(\theta_0-\theta')$ s large, though very nearly correct in the preceding applications in which the term kv, as we have seen, tends to reduce it and always to keep it small. We can, however, in this case, in which kv disappears, obtain an expression from (48) independent of any approximate development, from which the value of $(\theta_0-\theta')$ can be accurately computed. From this equation we get, putting $d\theta=0$,

$$(81) \qquad \mu^{\theta_0-\theta'}=\frac{a'K_0}{4Br\mu^{\theta'}}+r'e, \text{ or } \theta_0-\theta'=300 \log \left(\frac{a'K_0}{4Br\mu^{\theta'}}+r'e\right)$$

Where there is no solar radiation we have $K_0=0$, and

$$(82) \qquad \mu^{\theta''-\theta'}=r'e, \text{ or } \theta''-\theta'=300 \log r'e$$

in which θ'' is the shade temperature.

From (81) and (82) we get

$$(83) \qquad \mu^{\theta_0-\theta''}=\frac{a'K_0}{4Br\mu^{\theta''}}+1$$

which gives the relation between θ_0 the mean temperature of the body, and θ'' the temperature which it would have in the shade under the same circumstances.

The value of K_0 here is the mean normal intensity of solar radiation, undiminished by passing through an atmosphere where it can be expressed in a function of the form of (47), which can be done in this case for the annual mean and inequalities of annual period and its submultiples, as in (14), § 62, the values of the amplitudes and epochs being contained in the accompanying table. Since the sun shines on the body only one-half the time, the value of K_0 is one-half that of the solar constant, say 1.1. If we suppose the earth's surface to have a maximum radiating power we have $a'=r$, and for most radiating surfaces the difference perhaps is very small. For a body suspended near the earth's surface we have by (29), or from (70) by putting $p=1$, as we must in this case, $e=0.5$. If the earth's surface has a maximum radiating power we also have $r'=1$. With these values of K_0, r, and e, and the assumed value of $\theta'=0$, and putting $a'=r$, we get from (81) with the numerical value of B in § 76, $\theta_0-\theta'=-39°$. This result shows that the mean temperature of a spherical body near the earth's surface, supposed here to be kept at the temperature of freezing since we assumed $\theta'=0$, would be 39° below this temperature.

If there were no solar radiation we should have from (82) in this case $\theta''-\theta'=-90°$, if the law of Dulong and Petit, upon which these expressions depend, can be extended to so low a temperature.

If the body were suspended between ranges of mountains so that only two-thirds of the sky hemisphere were visible, we should then have $e=0.667$, and consequently the value of $(\theta''-\theta')$ greater, or the difference between θ' and θ'' less. If, however, the earth's surface had a small radiating power r', then the temperature of the body would differ from that of the earth's surface still more.

If the body were placed at such an altitude above the earth's surface that the solid angle subtended by it would be considerably less than a hemisphere, then by (29) we should have $e < 0.5$, and consequently the temperature of the body would differ still more from that of the earth's surface. A few miles, however, would give, contrary to what might be thought, scarcely any sensible difference for the temperature at which the body would stand.

If, in the case of no atmosphere and no solar radiation, a small body were placed in the focus of a parabolic partial inclosure of polished silver, subtending a solid angle of half a hemisphere, as in Fig. 5, § 73, the temperature of the body would be still much lower than that of the inclosure on account of the small value of r' for polished silver. (82) in this case gives no consistent result, because the law of Dulong and Petit, upon which it is based, cannot be extended to so low a temperature.

From (80^1) we readily see, from mere inspection, the general tendency which the various conditions of radiation, absorption, capacity of the body for heat, and the period of the inequality have upon the relation between the amplitude K of the inequality in the intensity of solar radiation, and the amplitude A, of the corresponding inequality of temperature depending upon it, and when the quantities entering into the expression are known their effects can be computed. If the body is so large, or the period of oscillation so short, as to make the value of $(Cu)^2$ very large, then A is small for any given amplitude of inequality in intensity of solar radiation, and is very nearly in proportion to the absorbing power a', since then the term in the denominator containing r, which is generally very nearly equal to a, becomes small in comparison. Hence large bodies with small absorbing power are very little affected by inequalities in intensity of solar radiation of short period. If, however, the value of $(Cu)^2$ is very small in comparison with the conjugate term, as in the case of a very small body or long period of temperature oscillation, then A is very nearly proportional to K.

In diurnal inequalities of solar intensity the value of u is 365 times greater than in the case of annual ones, and hence the value of $(Cu)^2$ is very great in comparison unless there is a corresponding decrease in the size of the body and value of C.

Retard in the variations of temperature.

91. From (80_2) it is seen that the greater the capacity for heat or the shorter the period the greater is the epoch ε; for the capacity of the body for heat C, being as the mass, increases in a greater ratio than the surface of the body s in the denominator of the expression. It is also inversely as the radiating power r of the body, and hence ε is largest in bodies of small radiating power, all other circumstances being the same.

Where the conditions are not such as to make the value of ε by (80_2)

so great that the arc cannot be used instead of the tangent, we get in the case of a sphere from (66), by putting $kv=0$, and for B its numerical value 1.146,

$$R=\frac{D\varDelta c}{.0528r\mu^{\theta'}} \tag{84}$$

in which

D=the diameter of the sphere;
$\varDelta$=its density;
c=its specific heat.

In the case of the blackened bulb of a mercury thermometer 1^{cm} in diameter, in which $r=1$, $\varDelta=13.6$, and $c=.033$, we get with $\theta'=30^\circ$, in which case $\mu^{\theta'}=1.26$

$$R=\frac{13.6\times 0.033}{.0528\times 1.26}=6.8$$

for the time in minutes by which such a bulb, where the temperature is increasing or decreasing, is retarded in arriving at the temperature of static equilibrium which the conditions would give for the solar intensity of radiation at the time; and this would be true not only in the case of one of the inequalities of (47) into which the varying intensity may be resolved, but approximately so for the sum of the effects, provided the values of ε by (80_2) in any of them does not become very large, as they would in very abrupt changes of intensity of solar radiation.

If the bulb were silvered, in which case we should have very nearly $r=.03$, it is seen from (84) that we should have this retard about 33 times greater, amounting to nearly 4 hours. If two thermometers, therefore, of the same size, one blackened and the other silvered, were exposed side by side to a varying intensity of solar radiation, with components of diurnal periods and its submultiples, they would at times indicate very different temperatures, for the silvered one would lag far behind the blackened one, and both behind one indicating the true static temperature. And it is readily seen that this also would be the case if the one thermometer bulb were much larger than the other, but the same in other respects—the larger one would lag behind the smaller one.

If we had a sphere 25^{m} in diameter of the specific heat by volume of water, we should have from (84), with $\theta'=15^\circ$,

$$R=\frac{2500}{.0528\times 1.122}=42160 \text{ minutes,}$$

or very nearly a month for the amount of the retard. Of course this would not be applicable to a diurnal inequality; but for an annual period, on account of the small value of u in this case, we should have by (80_2) the value of ε equal about 30°, in which the arc is approximately equal to the tangent, and so (81) is approximately applicable in this case. The temperature of such a body at any season of the year would be very nearly the same as that of an ordinary thermometer, with black bulb, one month before, and hence would be much less during the spring and much greater during the fall. In this case the amplitude A of temperature oscillation, given by (80), in comparison with that of a small-bulb thermometer in which $(Cu)^2$ sensibly vanishes in comparison with $(.0077\ rsB\mu^{\theta'})^2$, would be as cos ε to unity. Hence there would not only be a lagging behind in the case of the large body, but its whole range of temperature oscillation would be less.

Where there is an atmosphere surrounding these bodies, and especially where it is in motion, we have seen that the effect of it is to reduce the temperature of all the bodies to its own temperature whatever that may be, but still, even in this case, there is the tendency to assume the temperatures which they would have in the case of no atmosphere, and they do in some measure unless there is a strong ventilation.

Static temperatures.

92. Where the intensity of solar radiation can be regarded as constant, as at times of maxima and minima, or where variable, if the rate of change is so slow, or the body so small, that the first member of (44) sensibly vanishes, we get, in case of a vacuum or no atmosphere

$$(85)\qquad \mu^{\theta-\theta'}=\frac{\rho I}{B\mu^{\theta'}}+r'e,\ \text{or}\ \theta-\theta'=300\ \log\left(\frac{\rho I}{B\mu^{\theta'}}+r'e\right)$$

in which

$$(86)\qquad \rho=\frac{a'S}{rs}$$

and in which I has its value in (94) or more accurately in (101), Chap. 1, and S must be taken as defined in § 72.

From (82) and (80) we get

$$\mu^{\theta-\theta''}=\frac{\rho I}{B\mu^{\theta''}}+1$$

which is substantially the same as (81) in the case of mean solar intensity, and shows the relation between the static temperature of the body and its shade temperature under the same circumstances.

It is seen from these expressions that the temperature θ of the body depends not only upon the normal intensity of solar radiation I and the temperature of the inclosure θ', but likewise upon the value of ρ, and hence, by (86) upon the relation between the radiating power of the body r and its absorbing power for solar heat a', and upon the ratio between S and s, and therefore upon the shape of the body. Any change in these relations which increases the value of ρ increases the static temperature of the body, and *vice versa.*

In the case of an adiathermic spherical body of maximum radiating and absorbing powers, or in any case in which we can assume that a' is equal r, we have by (86), since in a sphere $s=4S$,

$$(87)\qquad \rho=\frac{1}{4}$$

In a cylinder such that the area of the ends may be neglected in comparison with the whole surface, if the sun's rays fall upon it perpendicular to its axis, we have, when $a'=r$, as it does usually very nearly,

$$\rho=\frac{1}{\pi}=0.318 \tag{88}$$

If the body is in the form of a flat disk, such that the surface of the edges may be neglected in comparison with the whole, and the sun's rays fall perpendicularly upon it, since $s=2S$ in this case, we have, where $a'=r$,

$$\rho=\tfrac{1}{2} \tag{89}$$

If, however, the edge of the disk were toward the sun, the value of ρ would be very small.

If the disk has a radiating power which is different on the two sides, the value of r in (86) must be the mean of both. If the disk, therefore, had a maximum radiating and absorbing power on the side exposed normally to the sun's rays, and a radiating power on the other of polished silver, say 0.03, we should then have

$$\rho=\frac{1}{\tfrac{1}{2}(1+0.03)\times 2}=0.971 \tag{90}$$

But if the bright side were exposed normally to the sun's rays we should have

$$\rho=\frac{0.03}{\tfrac{1}{2}(1+0.03)\times 2}=0.0291 \tag{91}$$

If we had a horizontal and very thin disk, with equal radiating and absorbing powers on both sides, with the sun's rays falling upon it with varying angles of incidence at different times, we should have the relation of $S=\tfrac{1}{2}s\cos z$, in which z, as heretofore, is the sun's zenith distance. We should therefore have in this case, with $a'=r$,

$$\rho=\tfrac{1}{2}\cos z \tag{92}$$

If the value of a' should differ a little from that of r, as it may in some cases, the preceding values of ρ would be modified accordingly.

93. In the case of a bright thermometer bulb of glass and mercury the consideration of the effect of the diathermic envelope of glass surrounding the mercury, which has scarcely any absorbing power for the solar rays, but a very great one for the heat radiated by the mercury, comes in as a modifying circumstance. The absorbing power of the bulb, taken as a whole, for the sun's heat, which passes mostly through the glass is that of the mercury, the part absorbed by the glass being extremely small. Putting the part which is transmitted through the glass equal to 0.80, it being very nearly the whole heat which is not reflected, and the absorbing power of mercury equal 0.2, we get 0.80×

0.2=0.16 for the absorbing powers of the bulb. But the radiating power of the bulb is not that of a thick plate of glass, since radiation does not take place from the surface merely, but from some small depth within. Magnus found that the radiating power of a plate of glass one millimeter in thickness was only 0.64, while that of a thick plate is supposed to be about 0.8. The deficiency, however, in the radiation of the thin glass is in some measure compensated by the radiation of the mercury through the glass, and would be quite if the radiating power of the mercury were as great as that of the glass. The glass of a thermometer bulb being generally much less than one millimeter in thickness, the radiating power of the bulb is less than that of a plate of glass one millimeter in thickness, or 0.64, as was observed by Magnus. If we put it equal 0.5, and the absorption, as above, at 0.16, we shall then have from (86) in the case of a spherical bulb, in which $s=4S$.

$$\rho=\frac{0.16}{0.50}\cdot\tfrac{1}{4}=0.08 \tag{93}$$

In this it is assumed that the diameter of the sphere of mercury and that of the whole bulb are sensibly the same. Since S in this case is the great circle of the mercury sphere and s the surface of the glass, if the thickness of the latter were considerable we should have s considerably greater than $4S$, and consequently the value of ρ in (93) considerably less in this case. Where, however, the glass is very thin, a small increase of thickness would have the contrary effect, since in this case the increase of radiating power from the increased thickness of the glass would have a greater effect in increasing the value of ρ, than the decrease of S would have in decreasing it.

94. In the case of no intensity of solar radiation, by putting $I=0$ in (85), we get (82), which is the equation for determining the static temperature resulting from nocturnal cooling, which has already been considered in § 90. In the case of solar radiation, in which I does not vanish, (85) becomes the same as (82) if we put the mean intensity K_0 for the varying intensity I at any time; for where the body is so small that it assumes very nearly at each instant the static temperature, I can be regarded as a constant sensibly for the time. It is seen, however, from (83) and the applications following, that (85), in the case of ordinary diurnal variations of I, is strictly applicable in the case of very small bodies only unless it is near the time of maximum intensity about noon.

In the case of no atmosphere the static temperatures, and all such as can be so regarded for the time, as in the case of the mean temperatures in § 90, are reduced very much from the effect of incompleteness of inclosure depending upon the value $r'e$ in (85) in which e becomes in this case equal 0.5.

It will be interesting and instructive to make a few applications of (85) to small bodies of different shapes and radiating and absorbing powers, in order to see how much the static temperatures depend upon these circumstances.

If we suppose the earth's surface to stand at the temperature of freezing and to have the maximum radiating power, we shall have $r'=1$, $r'e=0.5$, and $\mu^{\theta'}=1$; and in the case of no atmosphere, $I=A$, equal, say, to 2.2. In the case of a sphere we have also by (87) $\rho=0.25$. With these values (85) gives

$$\theta=300 \log \left(\frac{0.25\times 2.2}{1.146}+0.5\right)=300\times -.0088=-2.6^{\circ}$$

The temperature of such a body, therefore, exposed to the full intensity of the sun's rays, would stand 2.6° below the temperature of the earth's surface. If we suppose the temperature of the earth's surface to be 30°, we have $\mu^{\theta'}=1.26$, and we then get

$$\theta-\theta'=300 \log \left(\frac{0.25\times 2.2}{1.146\times 1.26}+0.5\right)=-16.5^{\circ}$$

and hence $\theta=30^{\circ}-16.5=13.5^{\circ}$.

In the case of a cylinder with the rays falling upon it perpendicular to the axis, using the value of ρ in (88) in this case, and assuming $\theta'=0$, we get $\theta=13.6^{\circ}$.

With a thin disk, and $\theta'=0$, using the value of ρ in (89) in this case, we get $\theta=49.2^{\circ}$. If the disk were of polished silver on the side not exposed to the sun's rays, using the value of ρ in (90), we get $\theta=112^{\circ}$. But with the silvered side exposed to the sun's rays, using the value of ρ in (91), we get $\theta=-76.5$, a temperature 188.5° lower than when the other side is exposed to the sun's rays.

If we had a very thin horizontal disk, such that it would arrive sensibly at each instant to its temperature of static equilibrium, using the values of ρ in (92), (85) with $\theta'=0$, would give for the sun's altitudes of 15, 30, 45, 60, 75, and 90°, the corresponding temperatures respectively of -37.0°, -2.6°, 22.1°, 38.1°, 47.0°, 49.2°, while a small spherical body, we have seen, would remain stationary at -2.6°. These altitudes at the equator would correspond to the several hours of the forenoon or afternoon.

95. If the bodies were placed in a vacuum within a diathermic inclosure in the atmosphere, the temperatures would be somewhat different, both because of the weakening of the intensity of solar radiation in passing through the atmosphere to the bodies, especially where the sun is near the horizon, and also because the inclosure in this case would be very nearly or quite complete with reference to the radiations from the body, so that instead of $e=0.5$ in (85) we should have it very nearly equal to unity. We have seen, (70) and (74), that the atmosphere itself nearly completes the inclosure with reference to terrestrial radiations, and if with this is included the immediate inclosure of the vacuum, which may be diathermic in a high degree to solar radiation,

but yet transmits but little of the heat radiated from the body, it must add to the completeness of the inclosure for such radiations. The temperatures of the bodies, consequently, would stand but little below that of the air and immediate inclosure of the bodies at night when the sun does not shine on them, since with a nearly complete inclosure or $e=1$, (85) gives, when $I=0$, very nearly $\theta-\theta'=0$. On this account the temperatures of the bodies would all be greater, but the range of temperature depending upon the varying values of I in (85) from 0 to maximum, would remain somewhat the same as in the case of no atmosphere, except that the extreme value of I at the earth's surface, even for a vertical sun, falls considerably short of the value of the solar constant.

96. If, then, we had a number of small bodies of different radiating and absorbing powers, shapes, and positions, exposed near the surface of the earth, without an atmosphere to solar radiation, or in a vacuum within a diathermic inclosure, they would stand at any instant at very different temperatures, and the ranges of temperature, depending upon the changes in the intensity of radiation, or value of I, would likewise be very different in the different bodies. At night also, and during polar winters, the temperatures of such bodies would be very far below that of the earth's surface.

Where, however, there is an atmosphere, and such bodies are in contact with it, and especially where the atmosphere has motions relatively to the bodies, the temperatures of the bodies are all reduced, as has been shown in the case of large bodies, very nearly to that of the atmosphere; but still it is seen from the results above how great the tendency is for the bodies to assume temperatures which differ very much from one another and from that of the air surrounding them, and consequently, in order to obtain the temperature of the air by means of that of a body immersed in it, how important it is to have a strong ventilation of the body, since it is only by means of this that the temperatures of the two become sensibly the same, either by day or by night.

Black-bulb and bright-bulb thermometers in vacuo.

97. On account of the smallness of the bulbs usually (85) is approximately applicable in this case, and the more so the smaller the bulbs. If the glass inclosure, together with the earth's surface on one side and the atmosphere around and above on the other, can be regarded as a complete inclosure, we have $re=1$ in this case, and the temperature θ of the body, where there is no solar radiation, becomes the same as θ, the temperature of the glass inclosure, as seen from (82), which (85) becomes when $I=0$. But observation shows that both the black and bright bulbs during a clear night stand about 2.3° F. below the temperature of the air in which the inclosures are suspended, and therefore, unless the inclosures stand at a temperature as much below that of the air, which is not probable to the full extent, the bulbs do not in this case assume the temperatures of their inclosures, but are a little

lower, and hence the glass inclosures cannot be regarded as complete inclosures, and we cannot put in (85) $r'e=1$. And this may also be inferred from the experiments of Melloni, § 70, from which it is seen that a small part of the heat radiated from a body at the temperature of 100° passes through thin glass, and therefore some of the heat radiated from the thermometer bulbs *in vacuo* must pass through both the glass inclosures and through the atmosphere, when it is clear, into space, and hence they cannot form a complete inclosure. We cannot infer, however, from the result of his experiments, that this lack of completeness and its effect can amount to much, but still the temperature of the bulb, where there is no solar radiation, no doubt, stands at a temperature a little below that of the glass inclosure, and still more below that of the surrounding air, unless this inclosure is so ventilated as to keep it sensibly at the air temperature.

The inclosure is so nearly complete and the radiating power of glass so nearly equal to unity, that we can put in (85) in this case, $r'=1$, without sensible error, and we then get

$$\mu^{\theta-\theta'}=\frac{\rho I}{\beta\mu^{\theta'}}+1-m \tag{94}$$

in which m is a function of the form of (71) except that p in this case is the diathermancy constant of both the atmosphere and the glass inclosure, and consequently smaller than in the case of the atmosphere alone. The value of m is therefore smaller than in the case of a bulb exposed in the open air, as given in (74), or even in Pouillet's actinometer as given in (73). With one observation of $(\theta-\theta')$ at night (94), by putting $I=0$, will furnish an equation for determining its value. For a very cloudy atmosphere of course its value vanishes.

98. The value of m and of I being known in any special case, (94) is applicable in the case of either the black or the bright bulb. For the former we must use the value of ρ in (87), and for the latter some value approximately equal that given in (93). The latter, we have seen, varies considerably in different bright bulbs, and therefore the temperatures of different bright bulbs may vary considerably among themselves with the same intensity of solar radiation, while those of the black bulbs, if they have a maximum absorbing and radiating power, and they lose heat by radiation only and not by conduction up the stem, should all be the same. On account of the widely different values of ρ in the two cases, the corresponding values of $(\theta-\theta')$ are very different.

If in (94) we put $m=0.015$ instead of the value in (74), we get, by putting $I=0$,

$$\theta-\theta'=300 \log 0.985=-2^\circ$$

for the amount by which the temperature of the thermometer is less than that of the glass inclosure, or of the air where the inclosure is kept at the air temperature by ventilation. The value of $(\theta-\theta')$ in this case is entirely independent of the radiating and absorbing power of the body, and consequently the same for both black and bright bulbs.

If we suppose the intensity of solar radiation to be 1.6, which is about the intensity at the earth's surface of a vertical sun in clear weather, and put $m=0.015$, we get from (94), with the value of ρ in (87), supposing the temperature of the inclosure to be 30°, with which $\mu^{\theta'}=1.26$,

$$\theta-\theta'=300\log\left(\frac{\frac{1}{4}\times1.6}{1.146\times1.26}+0.985\right)=30.3^\circ$$

for the difference between the temperature of the black bulb *in vacuo* and the temperature of its glass inclosure. In like manner, using the value of ρ in (93), we get

$$\theta-\theta'=300\log\left(\frac{0.08\times1.6}{1.146\times1.26}+0.985\right)=9.3^\circ$$

for the difference between the temperature of the bright bulb *in vacuo* and the temperature of the inclosure.

If the temperature of the glass inclosure were assumed to be that of freezing, we should have $\mu^{\theta'}=1$ instead of 1.26, and hence the value of $(\theta-\theta')$ considerably greater. The same intensity, therefore, causes a greater difference between the temperatures of the thermometers and the inclosures in cold than in warm weather.

99. From (94) we get for the black bulb

$$\rho I=B\mu^{\theta}-(1-m)B\mu^{\theta'}$$

Distinguishing ρ and θ by means of a suffixed accent in the case of the bright bulb, we get in this case

$$\rho_{,}I=B\mu^{\theta_{,}}-(1-m)B\mu^{\theta'}$$

Eliminating I from these equations, we get

$$\mu^{\theta'}=c\mu^{\theta_{,}}+(1-c)\mu^{\theta} \tag{95}$$

in which

$$c=\frac{\rho}{(\rho-\rho_{,})(1-m)}=\frac{1}{(1-4\rho_{,})(1-m)} \tag{96}$$

by putting for ρ its value in (87) in the last form of expression.

Having given the observed values of θ and θ', the temperatures of the black and bright bulbs, (95), gives θ', the temperature of the glass inclosure and sensibly that of the surrounding air where there is ventilation. For this purpose, however, it is necessary to know the value of the constant c, which, by (96), depends upon the values of $\rho_{,}$ and m, neither of which is accurately known in any case, and both vary a little with different bulbs and glass inclosures, the former for reasons given in § 95, and the latter by (71), because inclosures of different degrees of diathermancy require a different value of p, and consequently of m. The latter also varies with different degrees of transparency of the atmosphere, but may be regarded as a constant for all kinds of clear weather.

The constant c, therefore, must be determined from observation for each pair of conjugate thermometers, the glass inclosures of which should be of the same transparency. This may be done from (95) with one observed value of θ, $\theta_{\prime\prime}$, and θ', which give an equation which determines c; but a great many sets of observations of θ, $\theta_{\prime\prime}$, and θ' may be made, which give as many equations of condition for determining c by the method of least squares.

Examples.

1. In the case of no atmosphere if a thermometer were suspended at night near the earth's level surface having a temperature $\theta'=15°$, and a maximum radiating power, at what temperature would it stand? (82) and (29.)

2. What would be its temperature if it were surrounded by mountains so that only two-thirds of the sky could be seen, and the temperature of the earth's surface were at the temperature of freezing?

3. What would be the difference between the thermometer and a parabolic partial inclosure, as in Fig. 5, § 73, of the radiating power 0.8?

4. What would be the amplitude of temperature oscillation of a spherical body 6 centimeters in diameter, of the density .8, specific heat 0.4, maximum radiating power, and $\theta'=20°$, corresponding to a diurnal inequality of the form of one of the components of (47) of amplitude $K=2.0$? (80.)

5. What would be the retard of such a body? (84.)

6. What would be the retard of a bright-bulb thermometer 0.8^{cm} in diameter, of radiating power of 0.75, and the temperature of the earth's surface, $\theta'=10°$? (84.)

7. With a solar intensity $I=1.5$, and temperature of the earth's surface $\theta'=15°$, what would be the static temperature of a spherical body of maximum radiating power *in vacuo*, supposing the inclosure complete? (85.)

8. What, under the same circumstances, would be the temperature of a bright-bulb thermometer? (85) and (93).

9. What would be the temperature of a very thin disk *in vacuo* of maximum radiating power, exposed normally to the sun's rays, of intensity 1.8, which would be about the intensity of a vertical intensity on the top of a very high mountain, with the temperature of the air and earth's surface at freezing?

10. What would be the temperature of a black-bulb thermometer in space away from the influence of the earth's radiations, but at the mean distance of the earth from the sun? (85) and (87), $e=0$.

11. What would be the temperature of the bright-bulb thermometer under the same circumstances, supposing the law of Dulong and Petit to hold for so low temperatures? (85) and (93), $e=0$.

IV.—TEMPERATURE OF THE ATMOSPHERE.

Decrease of temperature with increase of altitude.

100. The preceding conditions for determining temperature, and their applications in the determination of temperature under different circumstances, have been confined to the comparatively simple case of solid bodies near the earth's surface, either in the air or *in vacuo*. Those for the determination of the temperature of the atmosphere at different altitudes and in the different latitudes under the varying conditions of solar radiation, are contained in Professional Paper No. XIII of the Signal Service, but the whole subject is too complex to be treated here in so general a manner. We must, therefore, confine ourselves here to a less general and more simple view of the matter, and in this we can avail ourselves, to a certain extent, of the more simple relations and expressions with reference to solid bodies near the earth's surface.

The atmosphere being diathermic, any small part of it may be regarded as so much volume of a diathermic solid body of very small density, so that its temperature will depend mostly upon the same conditions as those which determine the temperature of the diathermic solid. The principal difference is the greater uncertainty with regard to the radiating power of a portion of the atmosphere and its absorbing power for the solar rays, and especially upon the relation between the two.

The static or the mean difference between the temperature of a body near the earth's surface and that of the surface itself is determined by the condition of (61). As any small portion of the atmosphere, however, may be regarded as floating in the surrounding parts and having no relative motion with regard to them, except in cases of great disturbances, the term kv in this case disappears, and as it may be regarded also as having so nearly the same temperature as the surrounding parts that there is no sensible conduction of heat to or from it, it is not necessary to substitute the constant c, § 81, in its place. The condition then to be satisfied in determining the temperature of such a portion of the atmosphere is that expressed by (81). Now we have seen that by this condition the temperature of a body is the greater the more complete the inclosure, and that the difference between the static temperature θ of the body and that of the earth's surface or partial inclosure θ', supposed then to be at the temperature of freezing, ranges, in case of no solar radiation, from 0 in the case of a complete inclosure to about -90° in the case of no atmosphere or at the top of the atmosphere, which corresponds to $e=\frac{1}{2}$ or a half inclosure. It is seen from (70) and (71) that the inclosure, which is nearly complete at the earth's surface, becomes rapidly less so with increase of altitude and so of the value of p, so that we have now arrived at the important conclusion, that in the case of no solar radiation, if the earth's surface were maintained at the temperature of freezing, the temperature of the strata of the atmosphere would decrease from a few degrees below freezing at the earth's surface to about $-90°$ at the top, or to very nearly that temperature at an altitude of only a few miles, since even there the inclosure, or value of e in (70), is very nearly reduced to 0.5. It is here supposed that the earth's surface is maintained at a constant temperature, and does not communicate heat to the atmosphere in contact, in which case, on account of a small lack in the completeness of the inclosure, it would stand a few degrees below the temperature of the earth's surface, just as the thermometer on the swan's-down or in Melloni's cups stand a few degrees below the temperature of the surrounding air and earth's surface.

Effect of the mean solar radiation.

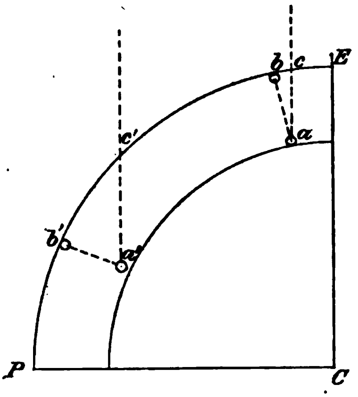
101. With the mean intensity of solar radiation, however, it is seen that the static temperature of a dark body, or any one in which the radiating power and the absorbing power for the sun's heat are equal, at or near the top of the atmosphere would be only 39° less than that of the earth's surface. On account of the uncertainty, in the case of air with regard to the relation between r and a' in (81) in the term depending upon the solar radiation of mean intensity K_0, we do not know what effect solar radiation would have in raising the temperature of the air, but we know that the absorbing power of the air for the sun's heat is considerably less than its radiating power, and therefore that the effect upon air is much less than upon a body of maximum radiating and absorbing power, or of any one in which $a'=r$. At the earth's surface also the temperature of the air would stand considerably higher than that in the case of no solar radiation, but not so high in general as the temperature of the earth's surface, on account of the smaller value of a' with regard to r in the case of the atmosphere. In either case, therefore, both of solar radiation and without any as at night and during the polar winters, in the static mean temperatures of the different strata, the difference between the temperature at the earth's surface and at the top is at least about 90°, so far as this depends upon the conditions of radiation, absorption, law of cooling, &c., without any consideration of the effects of conduction of heat, or convection in ascending and descending currents, of evaporation and condensation, and of cooling by expansion as the air rises, or heating by compressions as it descends and comes under a greater pressure.

Since the inclosure is rendered less complete and the value of e by (70) less, somewhat in proportion to the quantity and not the depth of atmosphere above any body under consideration, and so likewise the weakening of the solar intensity, whether the rays fall perpendicularly or obliquely upon the earth's surface, the decrease of temperature with increase of altitude must be rather in proportion to the mass of air left below than in proportion to the increase of altitude, and hence the rate of decrease with increase of altitude must be much greater below than above, as it is shown from observation to be. A decrease of, say, 100° through a homogeneous atmosphere of 8,000 meters of altitude would be one degree for each 80 meters, or 1.25° per 100 meters. This below would cause even dry air, § 38, to be in a state of unstable equilibrium, but above, as the air becomes rare, the rate would be a rapidly decreasing one.

There is also another effect which should be taken into consideration here, as in the case of solid bodies in the atmosphere, where there is a considerable depth of atmosphere between the body or portion of atmosphere under consideration and the earth, since this intercepts some of the earth's radiations to the body, and consequently its temperature is less than it would be if there was no intervening atmosphere. As there is no such effect on the air near the earth's surface, the rate of decrease of temperature with increase of altitude is therefore greater from this cause than that which has been determined from the preceding conditions, in which this has not been taken into consideration.

The effect of solar radiations is to cause a greater decrease of temperature with increase of altitude in the equatorial than in the polar regions of the earth. If we consider any small portion of air *a* near the equator *E*, and a similar one, *a′*, near the pole *P*, in the figure, but both near the earth's surface, it is seen that the solar intensity on *a* is stronger than on *a′*, because the rays have to pass through a distance of atmosphere *ca* in the one case less than *c′a′* in the other, and consequently the effect on the temperature of the former is greater than upon that of the latter. But with regard to similar portions of air *b* and *b′* on the same latitudes at or near the top of the atmosphere, the solar intensity is sensibly the same, and consequently the temperatures of both are increased the same amount by solar radiation. If, therefore, the temperature of *a* is increased *more* by solar radiation than that of *a′*, but the temperature of *b* is increased only *as much* as that of *b′*, then the tendency of solar radiation is to make the difference between the temperature of *a* and *b* greater than the difference between *a′* and *b′*, and consequently to cause the rate of decrease of temperature with increase of altitude greater in the equatorial than in the polar regions. There is a greater tendency, therefore, to a state of unstable equilibrium in the equatorial than in the polar regions, and hence there are more cyclones and tornadoes in the former than the latter. In high latitudes it is said that not even cumulus clouds are ever seen, so that it would seem that there the atmosphere is never in the state of unstable equilibrium.

FIG. 8.



All the preceding results have been deduced in a more general manner in Professional Paper of the Signal Service, No. XIII.

Modifying causes.

102. It is now important to consider the various causes which modify the preceding deductions from the conditions alone of radiation, absorption, &c. The evaporation of water cools the air near the earth's surface and in the lower strata, and the condensation of the vapor above increases its temperature there; and thus the great difference of temperature between the lower and upper strata, which would result from the conditions of radiation and absorption alone, is decreased somewhat by this process, especially in the lower part of the atmosphere, where the rate of decrease of temperature with increase of altitude was found to be the greatest, and where condensation as well as evaporation mostly takes place. Where saturated vapor ascends, the rate of decrease is as given in Table XIII, but in all places where the air descends, the tendency is toward the rate of 0.98° per 100 meters, though on account of the very slow rate generally of descent, this rate is only approximately reached.

As there is more evaporation in the equatorial than in the polar regions, the tendency from this cause to equalize somewhat the temperatures above and below is stronger in the former than the latter. From Table XIII it is also seen that the rate of decreasing temperature in ascending currents of saturated air is less in the equatorial regions where the air has a high temperature than in the polar regions of low temperature, so that this also tends to equalize the greater rate in the former and smaller rate in the latter arising from conditions of radiation, absorption, &c.

But the greatest of all the modifying causes of the temperature of the atmosphere is undoubtedly the contact of the atmosphere with the earth's surface. The whole atmosphere is comparatively a thin stratum only surrounding the earth, and from contact with its surface must partake in a great measure of its temperature, whatever the tendency may be to assume, in accordance with the conditions of radiation, absorption, &c., a higher or a lower temperature. Hence there is little difference between the mean temperature of the earth's surface and that of the air in contact with and near it, so that the actual temperature of the atmosphere in different latitudes depends more upon that which the earth's surface assumes, under the conditions determining its temperature, than upon any relation between its radiation and absorption, and the varying solar intensity as it affects directly the temperature in different latitudes. We have seen that from these conditions alone the temperature at the top of the atmosphere would be the same in all latitudes, as at *b* and *b′* in the preceding figure, though differing considerably near the earth's surface. But on account of the great differences between the temperatures at different latitudes of the earth's surface, there are nearly the same differences in the temperatures in different latitudes of the upper strata of the atmosphere.

It is also from contact with the earth's surface that the mean temperature of the air differs at different longitudes on the same parallels of latitude, for it partakes of the same local variations of temperature which the earth's surface has in different longitudes from ocean currents and other causes. As there is more evaporation on the ocean than on land in proportion to the condensation, the air over the ocean is cooler and that over the land warmer from this cause, and this affects a little the relations between the temperature of the atmosphere on the ocean and on land, both in latitude and longitude.

Variations of air temperature.

103. The variations of the temperature of the atmosphere corresponding to the variations, diurnal and annual, of the intensity of solar radiation, so far as they depend alone upon the conditions of radiation and absorption, and are independent of convection and conduction, should be given approximately by (80), which is especially applicable to a body near the earth's surface. By this expression the less the absorbing power a' of the body for solar heat in comparison with its radiating powers r the less is the amplitude A of the temperature variation corresponding to any inequality of the intensity of solar radiation of which the amplitude is K. A small portion of the atmosphere is very much like a piece of glass which transmits most of the heat of the solar rays through it and absorbs but little. The value of a', therefore, in (80_1) when applied to the air, as in the case of a piece of glass, is small in comparison with r in the denominator, and hence the range of temperature oscillation is small, large changes in intensity of solar radiation causing only small ones in the temperature. We know in the case of the atmosphere that a' is much less than r, and consequently that the amplitude A of temperature oscillation corresponding to a given inequality of intensity of solar radiation is much less in the case of air than in that of a black-bulb thermometer *in vacuo*, in which case we have $a'=r=1$.

In the case of annual variations, on account of the small value of u in these, the term $(Cu)^2$ in the denominator in (80) is so small as to be insensible in comparison with $(.0077rsB\mu^{\theta'})^2$, so that ε sensibly vanishes and the temperature at each instant is sensibly that of static equilibrium, but in the case of diurnal variations it may be comparatively large, and so decrease very much the value of A, so that the amplitudes of diurnal temperature oscillation are no doubt much less than those of the annual corresponding to the same amplitude K of intensity of solar radiation in the two cases. It is probable, therefore, that the diurnal temperature oscillations of the upper strata of the atmosphere in the open air away from the influence of contact with the earth's surface are extremely small. This, however, cannot be indicated by observation on high mountain tops, since the warm heated air in contact with the warm sides of the mountain during the day is continually ascending and giving rise to a higher temperature than that which generally prevails at the same level in the open air at a distance from the mountain top.

V.—TEMPERATURE OF THE EARTH'S SURFACE.

Mean temperature of the whole globe.

104. The expression of the rate by which a cooling body in space, without a diathermic envelope and away from any sensible influence of the sun's radiations, loses heat, is given by (28), § 78, by putting $e=0$, since in this case there is not even a partial inclosure. At the mean distance of the earth from the sun the rate by which a spherical body receives and absorbs heat from the sun is $a'SA_0$ in which, as already defined, a' is the absorbing power of the body for solar heat, S the area of a great circle of the body, and A_0 the mean solar constant. Since the rate of losing heat in the absence of the sun's heat must be diminished by the rate at which it receives and absorbs solar heat, we have for the rate of losing heat in this case

$$-\frac{dH}{dt}=Bsr\mu\theta-a'SA_0$$

Regarding the atmosphere as a diathermic envelope which has a greater transmissive power for solar than for terrestrial radiations, we must, by § 80, put rm instead of r, and $a'n$ instead of a', and we then get

$$(1)\qquad -\frac{dH}{dt}=Bsrm\mu\theta-a'nSA_0$$

in which m and n have their values in (34) and (35), § 80, the former expressing the proportion of the radiations from each point of the surface of the body with all angles of emissions within the solid angle of a hemisphere, which escape through the envelope, and the latter the proportion of the solar radiations which penetrate through the envelope to the body with similar angles of incidence over a hemisphere of the body.

105. If the body were very small—as a thermometer bulb, for instance—or its conducting powers for heat infinitely great, θ might be regarded as a constant for all parts of the surface. The static temperature, then, or the mean temperature in case of a variable intensity of solar radiation A, is given by the preceding equation by putting the first member equal 0, and we thus get

$$(2)\qquad \mu^{\theta}=\frac{a'n}{rm}\cdot\frac{SA_0}{sB}=\rho\frac{n}{m}\frac{A_0}{B}$$

in which ρ has its value in (87). From (96), § 47, and (34) and (35), § 80, since m and n are exactly similar functions, both requiring to be integrated through a solid angle of a hemisphere, and differ only in the different values of α in (96), § 47, in the two cases, we get

$$(3)\qquad \frac{n}{m}=\frac{e^{-\alpha b}}{e^{-\alpha' b}}=e^{(\alpha'-\alpha)b}$$

in which the value of α for terrestrial radiation is distinguished by an accent, and in which $b=P:P_0$ is the barometric pressure in terms of the standard pressure of 760^{mm}. For sea-level, therefore, we have very nearly $b=1$ in all cases. Hence, it is seen that the more nearly α' approximates to α or b to 0, the more nearly $\frac{n}{m}$ approximates to unity. If either $\alpha'=\alpha$ or $b=0$, we have by (3) $\frac{n}{m}=1$. If, therefore, the atmosphere or any diathermic envelope had the same transmissive power for the terrestrial radiations as it has for the solar, we should have the same value for μ^0 in (2), and consequently for θ, the temperature of the body exposed to any intensity of solar radiation A_0, as we should have in the case of no atmosphere or envelope, the value of $\frac{n}{m}$ being unity in both cases. By putting $B=1.146$, and $A_0=2.2$, we therefore get from (2) and (87) § 92, for a sphere, in either of these cases, if we assume that the body has an absorbing power a' for solar heat equal to its radiating power r, as it has in case of a lamp-black surface, and very nearly in all cases,

$$(4)\qquad \mu^\theta=0.48$$

Where the body is so large, as in the case of the earth, that the surface, on account of the unequal distribution of solar radiation, does not assume the same temperature in all parts, then θ must be understood to mean the uniform surface temperature of the body which would radiate as much heat as it receives from the sun and absorbs. This would differ some from the mean surface temperature of the globe, and also from the mean static temperature of the whole mass, especially if the difference between the equatorial and polar temperatures were very great. The expression of (4) is the same as is obtained from (81) for a small body away from the sensible influence of the earth, in which case we must put $e=0$.

From (4) we get $\theta=-96^\circ$ for the approximate mean temperature of the earth if it had no atmosphere, if the law of Dulong and Petit, upon which the expression is based, can be extended to so low a temperature. And as the moon has no atmosphere this must be approximately its mean temperature.

106. In the case of an atmosphere surrounding the earth as a diathermic envelope (4) becomes

$$(5)\qquad \mu^\theta=0.48\frac{n}{m}$$

We know that the average temperature of the earth's surface is 15.4°, and with this value of θ we get from this expression

$$(6) \qquad \frac{n}{m}=2.343$$

With this value the relation between α' and α could be determined from (3) by putting $b=1$. It is readily seen that this condition gives a value for α', the transmissive power of the atmosphere for radiated solar heat, much greater than that of α, its transmissive power for terrestrial heat radiations, for any probable assumed value of α. This may be approximately determined for clear weather from (96), §47, by putting $p=0.75$, its approximate observed value at the earth's surface for the standard barometric pressure of $P=P_0=760^{\text{mm}}$.

The value of m in (6), as deduced from Pouillet's experiments on clear nights with an artificial sky, is only 0.05. With this value (6) gives $n=0.117$, which is a value very much smaller than that which would be given by (35) with a value of $p=0.75$. With the true value of n, therefore, and with a value of m depending upon the radiation of terrestrial heat radiations through the atmosphere, we should have a value of $\frac{n}{m}$ much greater than that in (6), and consequently a value of θ, or mean temperature of the earth's surface much greater than what it is if the atmosphere were clear. In cloudy weather the solar heat radiations are mostly absorbed and radiated at a considerable altitude above the earth's surface; but still the effect is felt indirectly at the earth's surface on account of the convective interchanging vertical currents between the upper and lower strata, and for this part by (34) and (35), §80, the ratio of n to m becomes more nearly equal to unity, since it approximates to and becomes unity at the top of the atmosphere. But we would still have for the whole amount of heat received and radiated a ratio of n to m greater than that in (6), and consequently a mean temperature of the earth's surface greater than that observed. All this, however, is based upon the hypothesis that all the radiated solar heat which penetrates through the atmosphere to the earth's surface has to be radiated directly back through the whole depth of the atmosphere into space, which is not the case. A great part of this heat is consumed in evaporation over all parts of the earth's surface, but especially upon the oceans, and is then conveyed to the upper strata of the atmosphere by ascending currents as latent heat and given out again in condensation at an altitude from which it is radiated out with much more facility than it is from the earth's surface. For all this part, therefore, of the heat received at the earth's surface the value of m in (5) is much greater than in the case of that which has to be radiated from the earth's surface through the whole depth of atmosphere, for which the value of m by (74), §87, is only 0.05.

The mean annual evaporation of the whole torrid zone, according to Dr. Haughton,[51] is 216^{cm}. The number of heat units required to evapo-rate one cubic centimeter of water at an average temperature is about 587. If we therefore suppose that the whole annual average evaporation over the whole surface of the earth is only 125^{cm}, we shall have

$$\frac{125\times 587}{365.25\times 24\times 60}=0.138$$

for the average amount of heat used in evaporation per minute on each square centimeter over the whole surface of the earth. If we put $n=0.5$ in (35), § 80, we shall have

$$\rho n A_0=\tfrac{1}{4}\times 0.5\times 2.2=0.275$$

for the average amount of heat received from the sun per minute on each square centimeter of the earth's surface. Upon this hypothesis of $n=0.5$, nearly half the heat received from the sun at the earth's surface is carried as latent heat up to higher strata of the atmosphere, from which it is radiated out with more facility, and for which, consequently, the value of m in (5) is much greater than it is at the earth's surface. In this way the ratio of n to m is reduced to that given in (6), which is required to give the observed mean temperature of the earth's surface.

If the whole atmosphere were without motion, or if the whole surface of the earth were dry, the small value of m in (34), § 80, which would result from such conditions, would give a value of $\frac{n}{m}$ in (5), which would give a value of θ, the mean temperature of the earth's surface, very much greater than it is.

107. In what precedes, no account has been taken of what is called *the temperature of space.* If there is any sensible temperature of that kind it would be taken into account by supposing the earth to be surrounded by an inclosure of maximum radiating power of very low temperature, such that the heat radiated and received by the earth would be equal to that received from the stars. This would be the temperature of space, and the cooling earth removed from the effect of solar radiations would finally reach this low temperature. In this case, instead of putting $e=0$, as has been done, we should have $e=1$, $r'=1$, but the value of θ' so small that the effect of the term would be very small.

If the black and bright bulb thermometers *in vacuo* furnish in any manner, however rude, a measure of the intensity of solar radiation, and the sun is not a remarkable exception amongst the innumerable heat radiating bodies of space, then, as has been shown, the temperature of space is extremely low, if not entirely insensible. This may also be inferred with considerable certainty from other considerations. If the stars generally are of the same nature as the sun, it is natural and reasonable to suppose that the ratio between the heat of the sun and that of all the stars combined is somewhat as the ratio between the light of the sun and that of all the stars, and consequently that the amount of heat received from the stars is extremely small, and its effect upon the mean temperature of the earth insensible.

108. Regarding the earth as a cooling body in space, which has not yet arrived at its static temperature, as we must, since in all parts there is found a temperature gradient conducting heat from the interior to the surface, then a small part of the observed mean temperature of the surface depends upon the rate with which heat is conducted to the surface from the interior; for this requires just so much greater rate of radiation from the surface, and consequently a higher surface temperature. The effect is precisely the same as if so much more heat were received at the earth's surface from the sun. It has been ascertained from numerous underground temperature observations at many stations in almost every part of Europe, and from the known laws of the conduction of heat, that the rate by which heat is received from the interior at the earth's surface is about 0.0001 of a calorie per minute. This, compared with the average rate over the whole surface with which heat is received from the sun, namely, 0.275, as is seen above, indicates that the effect of the internal heat of the globe upon the mean temperature of its surface is very small. It has been shown in Professional Paper No. XIII that it increases the surface temperature at the equator 0.039°, but that the effect in the polar regions would be a little greater.[52]

109. The average rate of increase of temperature with increase of depth from observation is approximately 1° F. for each 60 feet. This of course differs in different localities, since it must depend upon the conductivity of the strata for heat. With the average rate of increase observed, the heat at no great distance toward the center would be so great as to keep, even under great pressure, the mass of all rocks in a liquid state, and hence there must be a comparatively thin solid shell, inclosing a liquid interior, composing the greater part of the earth's mass. This theory is the most plausible, and is required to explain a great many geological phenomena which do not seem to admit of explanation in any other manner; but it is by no means generally accepted.

110. With regard to secular and long period changes of mean annual temperature, we have no series of temperature observations sufficiently reliable and extending through a sufficiently long period of time to indicate any gradual secular change in the mean temperature of the earth, but they suffice to show that there cannot be any considerable changes of that sort. This would require a corresponding change in the intensity of solar radiation, but it is not probable that there has been any such change sufficiently great to produce any sensible effect upon the earth's temperature within a period of many centuries.

It is now known that there are regular periodical changes of the sun's surface, which, it has been thought, might affect the intensity of solar radiation, and thus the mean annual temperature of the earth's surface. A number of very accurate discussions of long series of temperature observations have been made at different places on the earth with reference to the 11-year inequality of sun-spot areas, but, as yet, it has not been clearly shown that there are any corresponding inequalities in the mean annual temperature. It is, however, probable that there are inequalities of this sort too small to be detected amongst all the other changes, diurnal, annual, and abnormal, of the earth's temperature. Such inequalities, having a common origin in the sun's surface, would have to be synchronous and similar over all parts of the earth's surface.

Mean temperatures of latitude.

111. An expression which gives the mean temperature of any given latitude may be obtained from (28), § 78, in the same manner that (2) has been obtained, but instead of $\rho n A_0$, which expresses the mean normal intensity of the solar-heat radiations as they reach the earth's surface after having passed through the atmosphere, we must substitute the vertical mean intensity K_0 of the latitude in (47), § 83. In this way we get in this case, instead of (2),

$$(7) \qquad \mu^\theta = 0.8726 \frac{K_0}{m} = 0.8726 \frac{\mu}{m} J$$

in which J is the vertical intensity of solar radiation at the top of the atmosphere, as given at the bottom of Table XI, in terms of A_0, and μ is the ratio in which it is diminished after the rays have passed through the atmosphere. μ is with relation to the rays of all zenith distances, at any latitude and for all seasons of the year, what n in (2) is with relation to the rays from a given direction in space falling on a hemisphere of the earth at all angles of incidence. Hence, toward the equator μ is greater than n, but toward the poles the reverse, and $\frac{n}{m}$ is a medium of all the values of $\frac{\mu}{m}$ over the globe.

According to this expression the values of θ for the different latitudes depend mostly upon the corresponding values of K_0, which we have seen, (§ 67), are a maximum at the equator and become comparatively small toward the poles. It also depends somewhat upon the varying values of m for the different latitudes, for, although these are independent of temperature, yet they may be considerably affected by the different hygrometric conditions of the atmosphere in the different latitudes, and, as we have just seen, also upon the different amounts of evaporation and condensation. These would tend to increase a little the value of m in the warmer equatorial regions, where, also, K_0 is greatest, and thus tend, in some measure, to equalize the equatorial and polar temperatures, which differ greatly on account of the much greater values of K_0 in the former than the latter. Since there is considerable uncertainty in the values of both K_0 and m on account of the uncertainty in the effect of the atmosphere upon both the solar and terrestrial heat radiations in passing through it, we cannot determine by means of (7) the values of θ for the different latitudes, but we can only infer that as the values of K_0 are very much greater in the equatorial than in the polar regions, so must be also the temperatures or values of θ.

The value of μ in (7) is greater toward the equator than toward the poles, since the average zenith distance of the sun for all seasons of the year is less and, as we have seen, m is also greater, the value of $\frac{\mu}{m}$ perhaps does not vary much from a constant, but it is, no doubt, some greater at the equator than at the poles.

If we suppose $\frac{\mu}{m}$ to be constant and to have such a value as satisfies (7) on the parallel of 30°, where $\theta=20°$, and by Table XI, $J=2.2\times0.268=0.590$, we shall have it equal to 2.265. With this value (7) gives at the equator, where $J=2.2\times0.305=0.671$,

$$\theta=54°$$

for the mean temperature of the earth's surface there. But at the poles, where $J=2.2+0.126=0.277$, we get

$$\theta=-78°$$

for the mean temperature. This large difference of 132° between the equator and the poles would, no doubt, be considerably increased by taking into account the greater value of $\frac{\mu}{m}$ at the former than at the latter.

112. This enormous difference between the equator and the poles which would exist in case of an earth with entirely dry-land surface, without transfer of heat from equatorial to the polar regions by means of ocean currents, is very much diminished, in the real case of nature, from the effect of such currents. It is estimated by Dr. Haughton that the Gulf Stream alone carries out of the tropics into the North Atlantic Ocean more than one-twelfth of the total heat received by the northern half of the torrid zone. He also estimates the amount carried away by the Kurosiwo, or Japanese current, at two and a half times as much. Putting it at the same only as the Gulf Stream, then one-sixth of the whole heat received by the northern half of the torrid zone is carried to the northern polar regions. We cannot say exactly in what manner the heat is distributed—that is, how much is lost in any given latitude of the equatorial regions and how much is gained in the polar—but as the spaces between meridians become narrower and vanish at the poles, with a loss of one-sixth part at the equator, there must be a gain of about twice as much at the poles.

A loss of one-sixth part at the equator is equivalent to reducing the value of J from 0.671 to 0.559, a decrease of 0.112. Twice this added to the preceding value of $J=0.277$, would give $J=0.501$. With these new values of J, decreased at the equator and increased at the pole, we now get at the equator

$$\theta=31°$$

and at the pole

$$\theta=16°$$

But this small difference between the equator and the pole which we now obtain would be considerably increased by the decrease of $\frac{\mu}{m}$ from the equator to the pole, which in the preceding computations has been regarded as a constant.

It is seen, therefore, how great a difference of temperature there would be between the equator and the poles with a wholly land surface of the earth, and how this great difference is kept down to that actually observed by no unreasonable hypothesis with regard to the transference of heat from equatorial to polar regions by means of ocean currents, one which requires a transfer much smaller than that given by the estimates of Dr. Haughton.

As the southern hemisphere has more water surface than the northern, the interchange of water between the equatorial and polar regions, and the transfer of heat from the former to the latter, is a little greater than in the northern hemisphere, and consequently the tropical regions of the southern hemisphere are a little cooler and those of the higher latitudes a little warmer than the corresponding parts of the northern hemisphere. This, however, may not extend to the pole of the southern hemisphere, for if there is an antarctic continent, its temperature, from what has been shown, must be extremely low, as that of the northern polar latitudes would be, if the north pole were not surrounded by water which feels the influence of the Gulf Stream, and also, in some measure, that of the Japan current. Since both hemispheres receive the same amount of heat during the course of the year, theory requires that their mean temperatures should be very nearly the same, and this now seems to be the case, though formerly, before observations were had from the higher southern latitudes, the southern hemisphere was supposed to be the colder.

The preceding results are important in showing what immense changes in climate may have occurred in many places in geological past ages from changes in the distribution of land and water. If this were such as to cut off mostly the interchange of the warm waters of the equatorial regions with the cold waters of the polar regions, the difference would be nearly that above in the case of a wholly land surface. On the top of a high mountain, or on a very high plateau, the volume of m, § 80, is enormously increased, while there is a comparatively small increase in the value of μ. The mean temperature, therefore, of the mountain top or high plateau, as given by (7), is very much diminished. The value of $\frac{\mu}{m}$ on the parallel of 30°, which gives the true normal temperature of latitude, we have seen, is 2.265. If we suppose the value of m with relation to μ to be so increased by elevation as to make the value of $\frac{\mu}{m}$ only half so much, we then get from (7)

$$\theta = -70^\circ$$

for the mean temperature at that elevation.

From the computed values of m in § 80 for the different altitudes, we see that it may be increased more than one-half at an altitude much less than that of Pike's Peak. And its temperature by the conditions of radiation and absorption alone, as given in (7), would be very low. But this merely indicates that the tendency is to a very low temperature, but the continual ventilation of the mountain top tends to keep the temperature very nearly, but not quite, up to the general temperature of the open air in the strata at that altitude, which depends upon conditions different from those expressed by (7).

113. In the case of no atmosphere we have $\frac{\mu}{m}=1$. With this, and the value of $J=0.671$, as obtained above, we get from (7)

$$\theta=-70^{\circ}$$

for the mean temperature at the equator, in case of no oceanic currents to convey any portion of the heat received from the sun.

With the value of $J=0.227$, obtained in like manner, we get from (7)

$$\theta=-211^{\circ}$$

for the mean temperature at the poles under like circumstances, if the law of Dulong and Petit can be extended down to so low a temperature.

Both these results, falling so far below the observed mean temperatures at the equator and the poles, indicate the great influence of the atmosphere upon the surface temperature of the earth.

If the sun were confined to the equator we should have $J=0$ at the poles, and thus they would cool down to absolute zero, or at least down to the temperature of space, if this is a sensible quantity; for the heat conducted from the equatorial to the polar regions through the solid earth would be too small to produce any sensible effect at the poles. As the sun does not shine on the poles of the moon, these must have sensibly the temperature of absolute zero.

Variations of mean temperature in longitude.

114. If the surface of the earth were homogeneous, either all water or all land of the same nature, or in any case where the temperature depends on the intensity of solar radiation alone, and not upon heat received from oceanic or aerial currents, which may differ very much in different longitudes, there would be the same mean temperature in all longitudes on the same parallels of latitude. This, however, is far from being the case, especially in the northern hemisphere. The transfer of heat from the equatorial to the polar regions, by which temperature is decreased in the former and increased in the latter, takes place mostly in the oceans in the manner already explained, the part transerred by the atmosphere being comparatively small on account of the smallness of its mass and specific heat. The equatorial regions, therefore, where the ocean exists, are cooled down by this transfer of heat,

and the temperatures of the polar oceans to which the heat is transferred are much increased, while on land the tendency is to retain in a great measure the same great difference of temperature between the equator and the poles which would exist if the whole surface of the earth were land and there was no transfer of heat from the equatorial to higher latitudes, either by oceanic or aerial currents. This causes the observed higher temperatures on land than on the ocean in the former, and the reverse in the latter. On the ocean the tendency of oceanic circulation is to equalize the temperatures of lower and higher latitudes, and the more so the greater the interchange of warm equatorial water for the cold water of the polar seas. But this equalizing effect is not confined to the oceans, but is in some measure felt on the continents also on account of the westerly component of the wind all around the globe in the trade wind latitudes, and the generally easterly motion of the atmosphere in middle and polar latitudes, by which heat is transferred from the warmer continents to the cooler oceans in the lower latitudes and the reverse in the higher, so that the observed difference between equatorial and polar temperatures on land is not nearly so great as it would be if it were not for the equalizing tendency of the westerly and easterly currents of the atmosphere around the globe to reduce land and ocean temperatures to the same. There are, notwithstanding, great differences of temperature on the same parallels of latitude arising from oceanic interchange of warm and cold water, especially in the northern hemisphere. There is a difference of about 30° F. between the mean annual temperature of the northern part of the North Atlantic, north and east of Iceland, and that of the northeast part of Asia and the northern part of North America on the same parallels of latitude. In the northern part of the Pacific the mean annual temperature is not quite so great as in the North Atlantic, because the heat transferred into higher latitudes by the Japan current is not contracted into so narrow a space as that of the Gulf Stream is.

In the higher latitudes of the southern hemisphere, where the ocean prevails in all longitudes except perhaps near the pole, there is nearly the same uniform temperature all around the globe on the same latitudes, though there are considerable differences on the same parallels in the temperatures of oceanic currents, the effect of which on the air temperature is not quite smoothed down to uniformity by the prevailing west winds of these latitudes.

There is also another cause of disturbance of the uniformity of mean temperature in longitude where both land and water exist. The evapo ration of water on the ocean is greater and on the land less than the condensation of aqueous vapor in the air by the amount of water conveyed from the land into the ocean by the rivers. The evaporation of this on the ocean cools the surface, and the condensation of it on the land warms the atmosphere there, so that the tendency is to decrease a little the temperature of the ocean and to increase a little the temperature of the atmosphere on land, and indirectly the temperature of the land surface. If, in the northern hemisphere, there were no oceanic currents the oceans would still be a little cooler than the continents from this cause.

Annual and diurnal variations.

115. For each inequality in the intensity of solar radiations expressed in the form of (47), § 83, there is a corresponding inequality in the temperature of the earth's surface of the form of one of the components of (50), § 83, but the relations between the amplitudes K, and A, and the epochs k, and ε, of the two expressions is much more complex, since it depends not only upon the heat capacity of the strata heated, but likewise upon the depth and the different degrees to which the strata are heated, and so upon the conductivity of the strata for heat. The relations in this case between the amplitude K of any inequality of intensity of solar radiation and epoch k, and the amplitude and epoch A and ε of the corresponding temperature inequality are expressed as follows:

$$(8)\qquad \begin{aligned} A &= \frac{a'K}{\sqrt{(.0077\ mBr\mu^{\theta_0} + Cu\nu)^2 + (Cu\mu)^2}} \\ \tan(\varepsilon - k &= \frac{Cu\mu}{.077\ mBr\mu^{\theta_0} + Cu\nu} \end{aligned}$$

in which

$$(9)\qquad \begin{aligned} \mu &= \int e^{-x\sqrt{\frac{C}{2h}u}} \cos\left(x\sqrt{\frac{C}{2h}u}\right)dx \\ \nu &= \int e^{-x\sqrt{\frac{C}{2h}u}} \sin\left(x\sqrt{\frac{C}{2h}u}\right)dx \end{aligned}$$

and in which

C=the specific capacity by volume of the strata,
h=their thermal conductivity,
x=the depth below the surface,
θ_0=the mean temperature of the earth's surface.

These expressions are only approximate, being obtained, as in the case of (67), § 84, from a solution carried only to the first approximation, and are not applicable to the smaller inequalities with much accuracy, on account of the numerous small terms neglected in the approximate solution. The effect on the temperature of the earth of the small changes in the relative temperatures of the air and the earth's surface is also neglected.

With the value of ε from (8) the retard in this case is given by (65) § 84, or, when this is small, by

$$(10)\qquad R = \frac{C\mu}{.0077\ mBr\mu^{\theta_0} + Cu\nu}$$

The value of m in these expressions is that of (34) § 80, and hence depends upon the diathermancy of the atmosphere for terrestrial heat radiations, and becomes unity in the case of no atmosphere.

The preceding expressions of these relations have been obtained by the same general principles as those of § 84, but for reasons just stated the process is too complex to be given here in detail, but it may be found in Professional Paper No. XIII,[52] with some slight differences of notation. From a mere glance at these expressions, as here given, more can be learned of the variations of these relations with the variations of the different circumstances of radiating and absorbing power, conductivity of, and capacity for, heat of the strata heated, period of inequality, &c., than could be from many pages of description.

The remarks in § 85 with regard to the effects of changes of radiating and absorbing power, capacity for heat, period, &c., upon the relations of (67) are also applicable to (8) above, but here the relations are also affected by the factors μ and ν, which, from their expressions in (9), it is seen, depend not only upon period of inequality, and the capacity of the strata for heat, but likewise upon their conductivity for heat. The greater the value of h, the greater by (9) are the values of μ and ν, and by (8_1) the value of A, the amplitude of the temperature inequality corresponding to any inequality in intensity of solar radiation with amplitude K, and also, by (8_2) and (10), the less the epoch and amount of retard. This is especially the case when C and u in the terms of which μ and ν are factors, are large, but if these terms are small the relation between A and K becomes nearly independent of μ and ν, and of the conducting power h of the strata for heat.

Since m decreases with decrease of the transmissive power of the atmosphere for terrestrial heat radiations in a greater ratio than K does, which arises from the development of (20) § 67, in which p^{ϵ} depends upon the transmissive power of the atmosphere for solar heat radiations, by (8_1) the less the transparency the larger the amplitude of the temperature oscillations, both annual and diurnal, but this would be especially the case in the annual oscillations, since in these the value of u is so small that the terms in which it occurs are comparatively small. However much the rate by which heat is received at the earth's surface may be diminished, its temperature oscillations are not diminished on this account, provided the rate by which it is radiated out into space is diminished in a greater ratio, and the period of the oscillation is such that the temperature arrives at nearly a state of static equilibrium, which requires the period to be very long and u small. But if the period is short, and consequently u large, the term containing m becomes so small in comparison with the others in the denominator of (8_1) that A is then diminished as K is diminished—that is, with decrease of the transmissive power of the atmosphere for solar heat radiations.

Where, however, the conditions are such as to give a large range of temperature and consequently high temperatures of the earth's surface in midsummer and at midday, the atmosphere becomes in a state of unstatic equilibrium, in which the earth's surface is cooled, and the large range prevented, which would occur under other conditions.

Comparisons with observation.

116. The data which enter into the preceding theoretical relations are mostly too uncertain to give any exact quantitative results for comparison with observation, but these relations are all confirmed in a general way by observation, and may, therefore, be safely extended where we have no observations.

The following table contains the averages of the mean temperatures on all longitudes, and also those of the mean temperatures of January and July, called *normals of latitude*, for each of the parallels of latitude given in the first column:

Lat.	January.	July.	Mean temperature.		Residuals.
			Observed.	Computed.	
°	°	°	°	°	°
+90	....	......	....	—17.0	....
80	—31.9	1.0	—15.5	—15.8	+0.3
70	—26.5	6.9	—9.8	—10.2	+0.4
60	—16.9	13.8	—1.6	—2.2	+0.6
50	—6.0	18.6	6.3	6.5	—0.2
40	4.5	22.8	13.6	14.4	—0.8
30	12.9	26.6	19.8	20.4	—0.6
20	21.7	29.0	25.3	24.3	+1.0
10	25.9	28.4	27.2	26.4	+0.8
0	27.3	26.1	26.7	26.8	—0.1
—10	27.9	24.0	25.9	26.0	—0.1
20	26.6	20.8	23.7	23.8	—0.1
30	23.0	15.6	19.3	20.2	—0.9
40	17.6	11.1	14.4	14.9	—0.5
50	11.1	6.4	8.8	8.2	+0.6
60	3.6	0.0	1.8	0.9	+0.9
70	....	......	....	—5.8	....
80	....	......	....	—10.6	....
—90	....	......	....	—12.4	....

The mean temperature of latitude θ is expressed in a function of the polar distance φ of the form

$$\theta = \Sigma_s A_s \cos s\varphi \tag{11}$$

in which the most probable values of A_s, determined from the data in the table by the method of least squares, are

$A_0 = 8.5°$ $\quad A_1 = -1.75°$ $\quad A_2 = -20.95°$
$A_3 = -1.00°$ $\quad A_4 = -2.66°$

With these values the computed values of θ in the table are obtained from (11), with the residuals in the last column.

The computed temperatures for the poles, especially the south pole, far beyond the limits of the range of observations, cannot be regarded as being very accurate, and it is not to be inferred from the results that the south pole is warmer than the north pole.

From (11) we get

$$d\theta = \Sigma_s sA_s \sin s\varphi = 0$$

for the condition which determines the value of θ for the maximum of the normal temperatures of latitude, called *the mean thermal equator*. The value of φ which satisfies this condition with the preceding values of A_s, is 88° 12′, corresponding to a latitude of 1° 48′ N. Its true position, therefore, in most, if not all, longitudes, is north of the equator.

If both hemispheres were precisely homogeneous in every respect, both with regard to the nature of the surface and the transmissive power of the atmosphere for solar and terrestrial heat radiations, the terms in (11) corresponding to the odd values of s would disappear, and the value of φ satisfying the condition above would be 90°, and the thermal equator would correspond with the geographical one.

The principal cause which disturbs the symmetry in the temperature of the two hemispheres is the ocean currents. It is readily seen that if the southern ocean transfers more heat away from the tropical region of its hemisphere than the northern ocean does, as explained in §112, the mean thermal equator must be a little north of the geographical equator.

The value of A_0 is the mean temperature of a meridian, but the average or mean with regard to the surface of the whole globe is given by

$$\theta_0 = \frac{1}{4\pi}\int \theta d\sigma = \tfrac{1}{2}\int \theta \sin \varphi d\varphi = 15.67^\circ$$

In the integration of this the value of θ in (11) must be used and the integration extended from $\varphi = 0$ to $\varphi = 180^\circ$. If the integration is applied to each hemisphere separately we get $\theta_0 = 15.30^\circ$ for the northern hemisphere and $\theta_0 = 16.05^\circ$ for the southern hemisphere.[61]

The following computed most probable normal temperatures of latitude have been obtained by Dr. Hann in later researches,[62] in which more recent observations on high southern latitudes have been used, which give to the results more weight than those above for the southern hemisphere should have:

South latitude.	Year.	Warmest month.	Coldest month.
	°	°	°
0	26.0	27.1	24.9
10	25.9	27.5	24.2
20	23.4	25.7	21.2
30	18.9	21.7	16.1
40	13.0	16.3	9.7
50	6.5	10.2	2.8
60	0.3	4.4	−3.8
70	−4.8	−0.4	−9.1
80	−8.2	−3.7	−12.7
90	−9.3	−4.8	−13.9

From these normals of latitude for the year he obtains 15.4° for the mean temperature of the southern hemisphere, almost precisely the same as that above for the northern hemisphere, and there is, therefore, perhaps, no sensible difference between the mean temperatures of the two hemispheres.

117. In the case of the mean temperature of the globe we know that a larger proportion of the solar heat radiations are transmitted through the atmosphere than of radiated terrestrial heat, and therefore that in (3) we have $\alpha' > \alpha$, and consequently $n > m$, and the value θ given by (2), when applied to the case of mean temperature of the globe, should be greater than in the case of no atmosphere, in which case we have $m = n = 1$. So, instead of a mean temperature of −96°, as obtained in the case of no atmosphere, we have an observed mean temperature of 15.4°. This requires the relation between m and n to be that of (6). We have no theory or observations from which to determine directly this relation with any degree of accuracy, but we know that $n > m$, and that therefore the observed mean temperature must be greater than it would be in the case of no atmosphere, which is in accordance with observation.

118. It is seen that the observed difference between the mean temperature at the equator and the poles is only about 40°, while theory, § 111, gives at least 115° in case of a wholly land surface. But it has been shown that with a very reasonable assumption with regard to the amount of heat transferred from equatorial to polar regions, one considerably less than Dr. Haughton's estimate, this difference would be reduced to 15° if we assume that $\frac{\mu}{m}$ in (7) is a constant, but as it undoubtedly must have a less value in higher than in lower latitudes, its true value would probably give about the observed difference of 40° between the equator and the poles. The observed difference, therefore, harmonizes very well with theory, so far as we are able to judge, with the uncertainty in the amount of heat transferred from lower to higher latitudes by the interchange of equatorial and polar waters.

From the computed normals of temperature in the preceding table for the northern hemisphere and those of Dr. Hann for the southern, we get the following comparison of these normals for the two hemispheres:

	10°	20°	30°	40°	50°	60°	70°	80°	90°
Northern hemisphere ..	26.4	24.3	20.4	14.4	6.5	− 2.2	−10.2	−15.8	−17.0
Southern hemisphere...	25.9	23.4	18.9	13.0	6.5	0.3	− 4.8	− 8.2	− 9.3
Difference	+ 0.5	+ 0.9	+ 1.5	+ 1.4	0.0	− 2.5	− 5.4	− 7.6	− 7.7

These differences indicate that the southern hemisphere is a little colder than the northern in the lower latitudes up to 45° or 50°, and beyond this the reverse. For reasons already given these results near the poles cannot be regarded as being very accurate. A satisfactory explanation of this result has been given in § 112, and it is exactly what should be expected.

They indicate, however, as has been assumed in § 106, that a transfer of heat from lower to higher latitudes raises the temperature more in the latter than it decreases it in the former.

Since the annual mean of J is the same for both hemispheres, §59, the northern and southern hemispheres should have the same mean temperature, unless the value of a', the absorbing power of water for the solar rays, is greater than its radiating power, as it is found to be at least in some of the metals. In that case the value of ρ as used in (2) would be, § 92, greater than unity, and consequently the value of θ would be a little greater for a water hemisphere than for one entirely of land, and as the southern hemisphere has more water than the northern its mean temperature should be the greater, if there is no counteracting effect. This would be found in the greater evaporation of a water surface, since, as has been shown, the heat used in evaporation at the surface becomes free again at a considerable altitude where it is more readily radiated into space, and where consequently it has less effect upon the general temperature of the earth's surface and the atmosphere.

119. The following table contains the differences between the temperatures for every tenth degree of longitude and the normals of latitude, temperatures minus normals, called *the abnormals of latitude*, for each of the parallels given in the first column:

Table of approximate abnormals of latitude.

Lat.	LONGITUDES WEST.																	
	180	170	160	150	140	130	120	110	100	90	80	70	60	50	40	30	20	10
70	—4	—4	—3	—2	—1	—1	—3	—4	—5	—6	—6	—4	—1	2	3	5	6	8
60	0	0	1	2	3	3	0	—3	—6	—8	—8	—8	—4	0	3	5	9	9
50	4	3	3	3	4	3	2	1	—2	—4	—7	—6	—5	0	3	5	6	5
40	—2	1	1	0	0	0	0	1	2	1	—2	—3	—1	2	3	4	3	3
30	—1	—2	—3	—4	—4	—4	—3	0	2	2	1	1	2	2	3	2	2	2
20	—1	—2	—3	—3	—3	—4	—4	—3	—1	2	0	0	0	0	0	0	1	2
10	—2	—2	—2	—2	—2	—1	—1	—1	—1	0	0	0	0	0	0	0	0	1
0	—1	—1	—1	—1	—1	0	0	0	0	1	1	1	1	1	0	—1	—1	—1
—10	0	0	0	—1	—1	—1	—1	—1	—1	—1	0	0	1	1	0	0	—1	—1
—20	—1	—1	—1	—1	—1	—1	—1	—2	—2	—2	—1	0	1	1	1	0	—1	—1
—30	1	0	0	0	0	—1	—1	—1	—1	—1	—2	—1	0	1	1	1	1	1
—40	0	0	0	0	0	0	0	0	—1	—1	—1	—1	—1	0	1	1	1	1
—50	0	0	0	0	0	0	0	0	—1	—1	—1	—1	—1	0	1	1	1	1
—60	1	1	1	1	0	0	0	0	—1	—1	0	1	1	1	2	2	2	1

Table of approximate abnormals of latitude—Continued.

Lat.	LONGITUDES EAST.																	
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170
70	9	11	11	7	3	0	0	0	0	0	–1	–3	–6	–6	–6	–5	–5	–4
60	9	9	7	6	4	3	1	0	–1	–2	–3	–5	–7	–7	–6	–5	–1	1
50	4	4	2	1	0	0	–2	–2	–3	–4	–4	–5	–5	–5	–4	–2	–1	2
40	1	3	3	2	1	0	–1	–1	0	0	0	–2	–4	–5	–4	–3	–2	–1
30	2	1	1	1	2	2	2	2	1	0	–1	–2	–4	–3	–2	–1	–1	–1
20	2	2	3	2	1	0	1	1	1	1	1	1	0	–1	0	0	–1	–1
10	3	4	5	4	2	1	1	1	1	0	0	0	0	–1	–1	–1	–1	–2
0	0	1	4	4	2	1	0	0	0	0	0	0	0	0	–1	–1	–1	–1
–10	–1	2	3	3	2	1	0	0	0	0	0	0	0	0	0	0	–1	–1
–20	–2	1	3	3	2	2	2	1	1	0	0	0	0	0	0	0	0	–1
–30	0	1	1	1	1	1	1	1	0	0	0	0	–1	–1	–2	–1	–1	0
–40	0	0	1	1	1	1	0	0	0	0	0	0	–1	–1	–1	–1	–1	–1
–50	0	0	0	0	0	0	0	–1	–1	–1	–1	–1	–1	–1	–1	–1	–1	–1
–60	0	0	–1	–1	–1	–1	–1	–1	–1	–1	–1	0	0	0	–1	–1	–1	0

It is seen from this table that in the higher latitudes of the northern hemisphere the signs of the abnormals are positive over the oceans and negative over the continents, and the reverse in the lower latitudes, showing that in the higher latitudes the mean temperatures are greater on the oceans than on the continents, but that on the lower latitudes the temperatures of the continents are greater than those of the oceans on the same latitude. This is exactly in accordance with the theoretical deductions in § 114.

In the lower latitudes of the southern hemisphere the mean temperatures are greater than the normals on land and the reverse on the ocean, but in the high southern latitudes, where there is no land, there is nearly a uniform ocean temperature all around the globe.

It is also seen that the highest mean temperatures in high northern latitudes are not over the middle of the oceans nor the lowest mean temperatures over the middle of the continents, but on the eastern sides. In the North Atlantic the highest mean temperatures of the higher latitudes are on the eastern side, near Europe, and on the continent the lowest mean temperatures are on the eastern side of Asia, near the Pacific Ocean. The same is true with regard to the North Pacific and the North American Continent on the east of it. This arises from the general easterly tendency of the atmosphere, by which the greater warmth of the oceans is carried over the continents, but affecting mostly the western sides, as has been already explained in § 114.

120. The observed amplitudes and epochs of both the annual and diurnal inequalities of temperature are found to vary very much, not only on different latitudes, but on different longitudes on the same parallel, the differences in general being very great between land and ocean. The following table contains the approximate amplitudes (half

ranges) of the first and principal annual inequality of temperature for each tenth degree of longitude for each of the parallels given in the first column:

Approximate amplitudes (half ranges) of annual inequality of temperature.

Lat.	LONGITUDES WEST.																	
	180	170	160	150	140	130	120	110	100	90	80	70	60	50	40	30	20	10
70	18	17	17	19	20	21	20	20	20	19	19	17	16	13	9	8	7	6
60	11	11	11	12	13	16	18	20	21	21	19	16	13	10	8	6	5	6
50	6	5	4	4	5	7	10	13	16	17	16	15	14	9	6	6	5	6
40	4	5	4	3	2	3	7	9	13	14	13	11	9	6	5	4	4	6
30	3	3	2	3	3	3	4	7	9	9	8	7	5	3	3	3	3	5
20	2	1	2	2	2	2	3	4	5	4	3	3	2	2	2	2	3	4
10	1	1	1	1	1	1	2	2	2	2	2	2	2	2	1	1	1	1
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
—10	2	2	2	2	1	2	2	2	1	2	2	2	2	2	3	2	1	1
—20	3	3	3	3	3	3	2	2	2	3	3	3	3	3	2	2	2	2
—30	4	4	4	3	3	3	3	3	3	3	3	4	4	3	3	3	2	2
—40	3	3	3	3	3	2	2	2	3	3	3	4	5	5	5	4	3	2
—50	2	2	2	2	2	2	2	2	3	3	3	3	3	4	3	3	3	2
—60	1	1	1	1	1	2	2	2	2	2	2	2	2	3	2	2	2	2

Lat.	LONGITUDES EAST.																	
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170
70	6	6	9	12	14	15	14	15	16	19	22	24	26	26	26	25	24	21
60	6	8	11	13	15	16	18	19	20	22	24	26	27	27	25	22	18	11
50	7	9	11	12	14	17	19	19	20	21	20	20	19	19	18	13	10	7
40	7	8	9	10	11	12	14	14	15	14	14	15	16	14	11	9	7	6
30	6	7	7	9	9	10	11	11	11	12	12	12	12	10	8	6	5	3
20	4	5	5	6	6	7	6	5	4	4	4	4	4	4	3	2	2	2
10	1	1	0	2	2	2	1	0	0	0	1	2	2	2	2	2	2	2
0	0	1	2	2	2	2	1	1	0	0	0	0	0	0	1	1	1	1
—10	1	2	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2
—20	2	3	4	3	3	3	2	3	3	4	4	3	4	4	4	4	3	3
—30	2	4	5	5	5	5	5	5	4	4	4	3	4	4	5	4	4	4
—40	2	2	3	3	3	3	4	4	3	3	3	3	4	4	4	4	3	3
—50	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	2	2	2
—60	2	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2

The numbers in the preceding tables, with some corrections introduced, have been deduced[61] from Buchan's Isothermal Charts, by means of rude approximate interpolations, in which the local irregularities are mostly lost, and they do not therefore claim to be very accurate representations of local irregularities and are merely designed to show in a general way the effect of latitude and of continents in increasing the annual range of temperature.

It is seen from these numbers that the annual temperature inequality of high latitudes, especially on land, is very great, but becomes much smaller in the lower latitudes, and very nearly vanishes at the equator.

On the same latitudes also the inequality is much greater on land than on the ocean.

The theoretical expression of the amplitude of this inequality is given by (8_1) by putting for K the value of K_1 in (22), § 67, which is the amplitude of the first and principal component in the expression of the intensity of solar radiation. But the value of K_1 becomes at the top of the atmosphere equal to A_0C_1 in (14), § 62, but at the earth's surface is decreased by the absorption and reflection of the sun's rays in the ratio of 1 to p^ϵ at any given instant of time, and hence, on the average, the value of K_1 is considerably less than A_0C_1, especially for the high latitudes, where the sun's zenith distance on the average is greater than for low latitudes. The values of K_1, therefore, do not increase with increase of latitude in a ratio quite as great as C_1 in the table of § 62, but still it must become large in comparison with its value in the middle and lower latitudes. This gradually-increasing value of K_1 from the equator to the pole, somewhat as C_1 in the table referred to, gives an increasing value of the amplitude A in (8_1), and this explains the largeness of the numbers in high latitudes in the preceding table.

121. The reason of the greater observed amplitudes on land than on water on the same latitude, given in the preceding table, is seen in the expression of (8_1). This is due to the greater value of C and h in (8) and (9) for water than for land; for water has both a great capacity and great conductivity for heat. An increase of the value of C in (9) decreases the value of μ and ν, but not in as great a ratio as C is increased, so that with an increase of C there is an increase of the value of the factors $(C\mu)$ and $(C\nu)$ in the denominator of (8_1), and consequently a decrease in the value of the amplitude A; and hence this value must be less on the ocean, where C is large, than on the land, where it is comparatively small. But the principal effect is undoubtedly due to the greater value which we must give to h in (9) on the ocean than on land; and this not so much on account of the real conduction of heat from the superficial to the lower strata and the reverse as to its transfer by means of the almost constant agitation of the water by the winds. Although the law of the diffusion of the heat in this way may be somewhat different from that of conduction, yet the effect must be nearly the same as if the conducting power for heat were very much increased. A large value of h in (9) gives corresponding large values of μ and ν, and consequently small values of A in (8_1), especially where the circumstances are such as to give the terms which contain μ and ν as factors a large value in comparison with $.0077 mBr\mu^{\epsilon_0}$.

As in the case of the mean annual temperatures, as seen in the table of § 119, so in the amplitudes of the annual inequality of temperature, as seen in the preceding table, the change from the characteristic of ocean to that of a land temperature does not take place at once in high latitudes, but gradually and mostly in an easterly direction on account of the prevailing easterly direction of the winds in those latitudes. For instance, taking the parallel of 50° N. in the preceding table, the amplitudes gradually increase eastward from 7° on the meridian of Greenwich to 21° on longitude 90° E., and on the parallel of 60° N. from 6° on the meridian of Greenwich to 26° on the meridian of 120° E. Starting also with longitude 130° W. on the parallel of 50° N. on the west coast of North America, where the amplitude is 7°, they gradually increase across the continent to longitude 90° W., where the amplitude is 17°. In the lower latitudes the amplitudes are so small that this effect is not so noticeable, and the surface winds having no prevailing direction either east or west, the greatest amplitudes are found near the middle of the continents.

122. The following table contains the amplitudes and epochs, *A* and *E* in the form of (50), § 83, for a few stations in America[63], Europe, and Asia,[64] so arranged as to show the contrast between maritime and inland stations:

MARITIME STATIONS.

Place.	Latitude.	Longitude.	A_1.	ϵ_1.
	° ′	° ′	°	° ′
Vardö	70 22	31 7 E.	7.8	210 42
Nova Zembla			11.7	210 19
Petropaulovsk	53 0	158 48 E.	12.0	209 44
Palermo	38 7	13 22 E.	7.4	210 55
Lisbon	38 43	9 8 W.	5.7	207 32
San Francisco	37 47	122 28 W.	2.3	215 5
San Diego	32 42	117 14 W.	5.0	210 10
Astoria	46 11	123 48 W.	6.0	207 16

INLAND STATIONS.

Taschkent	41 19	69 16 E.	13.3	190 3
Nukuss	42 27	59 37 E.	16.0	190 31
Urga	47 55	106 51 E.	21.3	192 37
Jakutsk	62 1	129 42 E.	31.1	192 45
Fort Craig, N. Mex	33 36	107 0 W.	12.3	190 29
Saint Louis, Mo	38 37	90 12 W.	13.3	195 4
Cincinnati, Ohio	39 6	84 30 W.	12.7	195 48
Fort Snelling, Minn	44 53	93 10 W.	16.7	195 23

By comparing maritime with inland stations of somewhat the same latitude, as Vardö with Jakutsk or San Francisco with Saint Louis, it is seen that the amplitudes of the former are very much smaller than those of the latter. The average amplitude of the maritime stations is 7.2°, while that of the inland stations is 17.1°, although the average latitude of the former is a little the greater. If the true amplitudes on the open ocean, far away from land, could be obtained and compared with those of inland stations in the interior of continents far from the ocean, the contrast between the amplitudes of the former and the latter would be still greater.

123. The epochs in angle ε_1 in this table are reckoned from the beginning of the year. Those of the maritime stations, it is seen, are all considerably greater than those of the inland stations. The average of the former is 210.2°, while that of the latter is 192.8°, a difference of 17.4°, corresponding to a retard greater by nearly 18 days on the average for the maritime than for the inland stations. The explanation of this is found in (8_2), as that of the differences in the amplitudes was found in (8_1). It has been shown that the value of $C\mu$, which comes in the numerator of (8_2), is greater for the ocean than for the land, and there tan $\varepsilon - k$ must be greater in nearly the same proportion, for the term $Cu\mu$, which is likewise increased in the same proportion, is generally a very small term.

The value of k_1 in (8_2) and (47), § 83, is the same as that of c_1 in the table of § 62, which for the average of the latitudes in the table above may be put at 169°. We therefore get from (65), § 84, and (13), § 61, for the average retard of the maritime stations,

$$R = \frac{(210.2^\circ - 168^\circ)}{.09856} = 42.9 \text{ days}$$

and for that of the inland stations

$$R = \frac{192.8 - (168^\circ)}{.09856} = 25.2 \text{ days.}$$

These are the times by which the maximum of the annual inequality follows the time of maximum intensity of solar radiation at or near the time of the summer solstice, and for the middle latitudes the afternoon of the 20th of June on the average, § 62. Here, as in the case of the amplitudes, the difference in the retard would be still greater between stations far away from land, and those in the interior of continents far away from the ocean. It must be understood, in accordance with what is stated in § 62, that the preceding applications of (8) hold strictly only up to the polar circles. They may, however, be extended up to the parallel of 70°, or even further, without any sensible error.

Another circumstance which affects the epoch of the annual inequality of temperature in polar latitudes is the conversion of large quantities of water and vapor into ice and snow during the fall and early part of the winter, by which latent heat is given out and the setting in of winter retarded, and their conversion back again into water and vapor during the spring and early part of the summer, by which the setting in of summer is retarded. The effect is the same as the adding another annual component to the one depending upon solar radiation with its minimum late in the spring when the ice and snow is melting most rapidly and with its maximum late in the fall when the ice and snow is being formed most rapidly. The effect of this is to throw the maximum and minimum of the resultant inequality a little later than they otherwise would be. This effect is very observable in monthly averages of temperature observations in high polar latitudes.

124. It is seen from the table of § 62 that the amplitudes C_2 of the semi-annual inequality of the vertical intensity of solar radiation at the top of the atmosphere are very small in the middle latitudes, but are considerably greater nearer the equator. The same, therefore, must be the case with K_2 in (22), § 67, which is the amplitude of the same inequality modified and diminished by the rays passing through the atmosphere to the earth's surface. The amplitudes of the corresponding semi-annual inequality of temperature, therefore, as brought out by the harmonic analysis of monthly averages, are very much larger at and near the equator than in the middle latitudes, where they are usually found to be only a small fraction of a degree, if the observations are sufficient to eliminate all the abnormal irregularities. As in the case of the annual inequality, so in this, (8) cannot be applied within the polar circles. On account of the large value of C_2 in the table of § 62 in comparison with C_1 at and near the equator, there are there two maxima and two minima in the resultant of the two components, which occur near the times of the equinoxes and solstices.

Observation indicates that on high latitudes within the polar circles there is a considerable retardation of the coldest part of the winter.[66] The coldest month is often February, but more usually March. The intensity of solar radiation being here a discontinuous function with regard to the annual, as it is usually with regard to the diurnal, variation, the cold continues to increase during the long polar night, as during the ordinary night, until the reappearance of the sun's rays, which, in the case of the polar night, is February or March, according to the latitude of the place. Hence the coldest part of the winter on the average is about this time, as that of the ordinary night is at or a little before sunrise, though various abnormal disturbing causes may often change the time of either considerably. The observations on high polar latitudes are too few to determine accurately the normal time of maximum cold on any given latitude, but it undoubtedly is about the time of the reappearance of the sun's rays.

From this cause the harmonic expression of the annual variation of temperature is not so convergent as in the lower latitudes, and the second or semi-annual, as well as components of still a lower order, are large, just as in the case of the diurnal variation, the semi-diurnal, and other components are always considerable in comparison with the diurnal and principal component.

125. In the application of (8) to the diurnal variation of temperature we have usually only a rough approximation on account of the slow convergency of (21), § 67, and the imperfection of the solution upon which (8) depends, since it is carried only to a first approximation, in which terms of the order of the smaller components in the expression of J are necessarily neglected. We can, however, in an approximate way, make some important deductions from (8) with regard to the diurnal variation.

From what has been stated, § 67, the values of K_n, and especially of K_1 in this case, must in general decrease from the equator to the pole. Therefore, by (8_1), the amplitudes of the several components of diurnal variation must in general decrease likewise from the equator to the poles, where the surface strata are such that the values of C and h are the same, or nearly, as in the case of all land or all water. The following table contains the approximate extreme diurnal range of temperature for January and July of the places in the first column arranged according to latitude:

Place.	Latitude.	Longitude.	RANGE.	
			January.	July.
	° ′	° ′	°	°
Washington	38 54	77 3 W.	10.2	17.4
Madrid	40 25	3 42 W.	7.2	14.5
Nukuss	42 27	59 37 E.	6.1	13.6
Astrachan.	46 21	48 2 E.	3.8	5.8
Nikolaiev	46 58	31 58 E.	3.1	7.9
Nertschinsk	51 19	119 44 E.	6.9	10.7
Dorpat	58 23	26 43 E.	1.5	8.7
St. Petersburg ...	59 56	30 16 E.	1.0	6.6
Hammerland	60 13	19 48 E.	1.5	3.0
Vardö	70 22	31 7 E.	0.5	3.5
Felsus Bay	70 37	57 30 E.	1.3	3.0
Suchta Bay	73 57	54 48 E.	0.4	3.1

From this table it is seen that the extreme range at least of the diurnal inequality gradually decreases with increase of latitude, and that in very high latitudes they become very small, especially for the maritime stations. It is also seen that the ranges throughout are much smaller in January than in July, and the relative difference is especially large in the high latitudes, all of which accords with what has been deduced from (8_1).

It has been shown, § 67, that the values of K in (8_1), in the middle latitudes, are much greater in the case of the diurnal than in that of the annual variation. But, comparing land stations of the middle latitudes, it is seen from the preceding table that the amplitude of diurnal oscillation is only about 3°, while by the tables of § 120 and § 122 the amplitude of the annual variation, having a much less value of K, is about five times as great. This is because the value of u in the case of the diurnal variation is three hundred and sixty-five times greater than in that of the annual, and so the terms in the denominator of (8_1) containing u as a factor are very much increased, so that although K is greater in the diurnal variation of solar radiation, yet A, the amplitude of the corresponding temperature inequality, is much less. But the increase of the terms containing u as a factor is not proportional to the increase of u, since C and u enter into both (8_1) and (9) in precisely the same manner. The effect is precisely in the same ratio as that arising from an increase of C, which we have seen in § 121 is in a much less ratio.

According to (8_2) the value of $(\varepsilon-k)$ in the diurnal inequalities should be large, on account of the large value of u in this case. The observations which we have give the time of the maximum of the resultant of all the temperature components corresponding to each of those in (21), § 67, as usually occurring from two to three hours after noon. We cannot infer, however, from this what would be the value of $(\varepsilon-k)$, and consequently the retard, in the case of each component, especially since the expression is not very convergent and consists of a number of sensible terms.

126. In the diurnal, as in the annual inequalities of temperature, both the amplitudes and epochs may be considerably affected by the winds. If during the warmer part of the day at a maritime station, the wind in summer blows from a cooler ocean, or in winter from the colder interior of the continent, as often happens, the diurnal range of temperature is in both cases diminished. And if the maximum of the wind occurs late in the afternoon it not only diminishes the range but the epoch of the maximum. It may, and often does, happen from this cause, mostly where land and sea breezes prevail, that the forenoon is warmer than the afternoon, and thus the maximum occurs even before noon. A similar effect is produced on the tops of high mountain peaks. During the forenoon, before the mountain's sides and the air in contact becomes heated up, there is but little ascending current up the mountain, but in the afternoon the current becomes very strong, and the cooling of the ascending current from expansion becomes sufficient to cause the temperature at the top to be very much less than it otherwise would be, if not to make the temperature greater than in the forenoon. This is especially the case in very clear weather. For this reason the maximum temperature on mountain peaks is usually observed to occur sooner after noon than in the plains below.

Underground temperatures.

127. Poisson[25] has shown that when the temperature of the earth's surface can be represented by an expression of the form of (50), § 83, as it always may, the temperature at any depth x below the surface is represented by

$$(12) \qquad \theta'=\Sigma A'_s \cos\,(s\omega t-\varepsilon'_s)$$

in which, approximately

$$(13) \qquad A'_s=A_s e^{-x\sqrt{\frac{C}{2h}s\omega}} \qquad \varepsilon_s=\varepsilon_s+x\sqrt{\frac{C}{2h}s\omega}$$

in which the definitions C and h are as in § 115, but $s\omega$ here is equivalent to u in (9), ω here being a constant for all the terms.

It is seen from these expressions that the greater the depth x, and the values of C, s, and ω, and the smaller the value of h, the smaller is the amplitude A'_s of the temperature oscillation and the greater the value of ϵ'_s and consequently the later the time of the maximum temperature, and *vice versa*. In the diurnal variation in which the value of ω is very large, in comparison with what it is in the annual, A_s becomes small relatively to A' at a very much smaller depth than in the case of the annual variation.

From (13) we obtain the following practical expression of the relation between A' and A

$$\log A'_s=\log A_s-Bx \tag{14}$$

in which

$$B=\log e\sqrt{\frac{C}{2h}s\omega}=0.4343\sqrt{\frac{C}{2h}s\omega} \tag{15}$$

This, however, assumes that the strata are homogeneous and C and h constants at all depths, which is never strictly the case, especially near the earth's surface, where the specific heat and conducting power varies at different seasons and at different depths from rainfall. Hence, the relation of (14) does not agree very well with the observed relations, especially near the surface.

The following are the approximate observed values of A'_s and ϵ'_s in (12), as obtained by Wild,[66] for the annual variation of temperature at St. Petersburg and Nukuss, at the several depths given:

ST. PETERSBURG.

s	$x=1.52^m$		$x=3.02^m$	
	A'_{s1}	ϵ'_{s1}	A'_{s1}	ϵ'_{s1}
	°	°	°	°
0	6.75	....	7.22	....
1	7.45	234.0	4.32	265.2
2	1.55	63.4	0.67	112.4
3	0.59	183.6	0.09	236.6
4	0.26	333.1	0.11	28.8
5	0.20	74.4		

NUKUSS.

s	$x=2.8^m$		$x=4.0^m$	
	A'_{s1}	ϵ'_{s1}	A'_{s1}	ϵ'_{s1}
	°	°	°	°
0	14.49	....	14.03	....
1	3.66	250.9	1.96	281.3
2	0.06	262.8	0.04	325.0
3	0.06	65.4	0.04	138.3

In both these sets of results from observation there is a decrease of the amplitudes A_s, and an increase of the epochs in angle ϵ_s, with increase of depth. With the values of ϵ'_s the times of maxima from January 15, which is the era or origin of t adopted in these results, are given by (65) § 84, by omitting k', which vanishes in this case, and using the value of $u=si=0.9856$. This would give for the time of maximum from January 15, of the first component at St. Petersburg, at the depth of 1.52^m,

$$\frac{234.0}{0.9856}=237.4 \text{ days}$$

corresponding to September 9. At the depth of 3.02^m we should have

$$\frac{265.2}{0.9856}=269 \text{ days}$$

for the time, from January 15, of the maximum, corresponding to October 11, which is more than one month later than it is at the depth of 1.52^m.

The times of maxima of the *resultant* of all the components were found to be, at the depth of 1.52^m, August 26; and at the depth of 3.02^m, October 3.

It is seen from the preceding tables that the amplitudes of the several components of annual temperature oscillation are very much less at the depth of 3.02^m than at the depth of 1.52^m, at St. Petersburg, or at the depth of 4^m than at 2.8^m at Nukuss.

Herr Wild has found that at thirty places in Europe and Asia the depth at which the amplitude of the annual inequality becomes 0.01° varies from 15^m to 33^m, according to the nature of the soil. The average is 24^m.

The following are the values of the approximate amplitudes and epochs of the first inequality in (12), obtained by Wild,[66] for each month of the year, from the observations of the diurnal variation of temperature at Nukuss:

Month.	$x=0.1^m$		$x=0.2^m$	
	A'_1	ϵ'_1	A'_1	ϵ'_1
	°	°	°	°
January	1.50	245.6	0.56	293.0
February	3.17	244.6	1.25	292.9
March	2.41	244.4	1.09	293.2
April	3.78	238.1	1.79	287.9
May	4.73	232.0	2.25	285.1
June	5.26	232.7	2.36	286.7
July	4.71	234.1	2.09	287.1
August	5.03	232.9	2.15	286.3
September	5.25	232.7	2.25	283.8
October	4.36	232.6	1.86	282.8
November	2.94	241.5	1.10	287.8
December	1.57	246.7	0.62	292.5

In this, as in the case of the annual variation, the amplitude decreases and the epoch increases with increase of depth. By (13_2) the difference between the epochs ϵ, for the depths of $x=0.1^m$ and $x=0.2^m$ should be a constant for all the months unless there is an annual change in the values of C or h. In the preceding table the differences are greater in summer than in winter, which may be caused by the conducting power h of the strata being less in summer when the soil is dry.

By dividing the epochs in angle by 15° we get the epoch in hours from midnight of the maximum of the first and principal inequality in the temperature, as expressed in (12), since $\omega=15$ and $s=1$ for this component. This, for the month of July, makes the maximum occur at $3^h\ 37^m$ p. m. at the depth of 0.1^m, but at $7^h\ 8^m$ p. m. at the depth of 0.2^m.

From the preceding table it is seen that the amplitudes of the diurnal inequality diminish more than one-half between the depths of 0.1^m and 0.2^m. Herr Wild has found that the depth at which the diurnal amplitude becomes 0.01°, is about 0.9^m on the average, being, of course, greater in summer than in winter.

Nocturnal cooling of the earth's surface.

128. This, as in the case of the nocturnal cooling of bodies near the earth's surface, § 86, is usually understood to mean the cooling of the surface relatively to a thermometer suspended in the air four feet above its surface. As the earth's surface has a greater radiating power than the air, it cools faster, and soon after sunset, and even before, it has a lower temperature when the state of the atmosphere is such as to allow a part of the radiated heat to pass through into space. There is, however, a limit to the difference between the two temperatures, beyond which it cannot extend, and this is determined by the condition that the surface cannot cool down below the temperature of a complete inclosure from which it would receive as much heat as it receives from the atmosphere, since a body cannot cool down below the temperature of its inclosure. The temperature of such an inclosure is *the mean sky temperature.* The equation of conditions in this case may be deduced from that of (28), § 78, for the case of a spherical body within a partial inclosure, the atmosphere in this case being that inclosure. Putting

θ_s=the mean sky temperature;

θ_a=the air temperature four feet above the surface; we shall have, in (28), § 78,

$$r'e\mu^{\theta'}=\mu^{\theta_s}=\mu^{\theta_a+(\theta_s-\theta_a)} \tag{16}$$

With this value of $r'e\mu^{\theta'}$, since $a=r$ in this case, the equation gives for a unit of surface area or $s=1$,

$$\mu^{\theta-\theta_a}=\mu^{\theta_a-\theta_s}-\frac{1}{Br^{\mu\theta_a}}\frac{dH}{dt} \tag{17}$$

In this expression D_tH is the rate at which a unit of surface is losing heat. This at first is comparatively large, but it gradually becomes less as the upper strata of the earth cool, but it never entirely vanishes. Putting $D_tH=0$, (17) gives $\theta-\theta_a=\theta_a-\theta_s$. Therefore $\theta_a-\theta_s$ is the limit by the formula to which $(\theta-\theta_a)$ approximates, but can never quite reach, as D_tH becomes smaller.

129. On account of the complexity of the subject an expression of D_tH in terms of the thermal specific capacity and conductivity of the strata cannot be deduced here, but from the expression of D_tH in § 25, equation (49), of Professional Paper No. XIII, we get

$$\frac{dH}{dt} = -Csu\,\Sigma(\mu A\cos\varepsilon - \nu A\sin\varepsilon)\sin sut + Csu\,\Sigma(\nu A\cos\varepsilon + \mu A\sin\varepsilon)\cos sut \tag{18}$$

in which all the quantities under the sign Σ except u vary for each of the inequalities of which the characteristic is s in Poisson's expression of the surface temperature, (50) § 83. The expressions of μ and ν are given in (9), except u there represents su here. With these expressions in (18) it is readily seen that D_tH is a function of the diurnal range of temperature, which depends upon the values of A for the several components; upon the specific capacity, C; upon the conducting power of the strata, h; and upon the value of u, or the period of oscillation. Although μ and ν are functions of C, yet we have seen, § 90, that $C\mu$ and $C\nu$ increase with C, though not in proportion, and therefore the less the specific heat by volume, the less D_tH. It is also seen from (9) that the smaller h the smaller are μ and ν, and that they vanish when h vanishes. Therefore the less the thermal conductivity of the strata the less the value of D_tH. Where there is no diurnal oscillation of temperature the values of A in (18) for the several components vanish, and consequently the last term in (17). We then get the greatest effect or value of $(\theta-\theta_a)$. The temperature to which the surface falls during the night and its relation to the air temperature depends upon the temperature gradient vertically in the strata at sunset, and the smaller this is, the smaller is D_tH, as given by (18). When the day, therefore, has been cloudy, but the night about sunset becomes clear, the greatest effect should be experienced. Also, the less the value of u—that is, the greater the period of temperature oscillation—the smaller the value of D_tH, and consequently for the long polar night it must become extremely small.

The value of D_tH is negative during the night, and has its greatest negative value near sunrise. With a negative value we have $\theta > \theta_s$, but the smaller D_tH the more nearly the temperature of the earth's surface θ coincides with the mean sky temperature θ_s, or the more nearly the difference between the earth's surface and the temperature at the height of four feet, $\theta-\theta_a$, coincides with the difference $\theta_a-\theta_s$). It is also seen from (17) that the greater the radiating power r the more nearly the last term vanishes, and consequently the more nearly $(\theta-\theta_a)$ approximates to $(\theta_s-\theta_a)$.

The value of $(\theta_a-\theta_s)$ depends upon the diathermancy of the atmosphere, and vanishes when it is very cloudy, since we then have a perfect inclosure, and consequently θ_a as well as θ equal to the sky temperature θ_s. From the average for a number of clear nights Pouillet obtained, by means of his actinometric observations and artificial sky, § 87, a value of 16° for the difference between the temperature of a thermometer four feet above ground and what he calls the zenithal temperature, which is the mean sky temperature for two-thirds of a hemisphere. As the remaining third lay around near the horizon, in which the sky temperature is very nearly that of the thermometer, the value of the mean sky temperature would be a very little more than two-thirds of 16, say 12°. This, then, according to the observations of Pouillet, is the limit beyond which the value of $(\theta-\theta_a)$, usually called *nocturnal radiation*, cannot go. It is not to be supposed that with the changing temperatures and changing relations between the temperature of the atmosphere and earth's surface, the zenithal temperature of Pouillet, or the mean sky temperature, has the same relation at all times to the temperature of the air at the height of four feet; but after the temperatures have arrived near their minima, and are nearly in a state of static equilibrium, as they are most of the night, this relation at sea-level, and for the same degree of diathermancy of the air, must be nearly the same, as Pouillet's observations indicate. With increase of altitude and diminution of atmosphere above, of course the inclosure becomes less perfect, and by (16) the less the mean sky temperature θ_s, and consequently the greater the value of the constant, so regarded, $\theta_a-\theta_s$ in the expression of (17), and, therefore, of $(\theta-\theta_a)$.

From what precedes, therefore, nocturnal radiation, so called, depends (*a*) upon the diurnal temperature oscillation, (*b*) upon the diathermancy of the atmosphere for terrestrial radiations, (*c*) upon the radiating power of the earth's surface, (*d*) upon the thermal capacity of the upper strata of the earth, (*e*) upon their thermal conductivity, (*f*) upon the altitude above sea-level. The observed effect $(\theta-\theta_a)$, therefore, is not even a relative measure of radiation, inasmuch as from the manner in which r enters into the expression of (17) there is no proportionality between $(\theta-\theta_a)$ and r, even when all the other circumstances remain the same. It is simply a measure of the combined effect of all the conditions.

130. Where the radiator is a body not forming a part of the earth's surface but is in contact with it or immersed in the stratum of atmosphere close to the earth's surface, its temperature, however great its radiating power, cannot differ much from the temperature of the surface and the stratum of air in contact, and the effect observed depends almost entirely upon the conditions which determine this in case of the earth's surface.

By (17) the effect of nocturnal cooling $(\theta-\theta_a)$ is increased with increase of r and with decrease of D_tH. But we have seen that D_tH is decreased with decrease of thermal capacity and thermal conductivity of the earth's upper strata and by smallness of diurnal temperature oscillation. It is also increased with increase of diathermancy of the atmosphere and of altitude, both of which increase the limit $(\theta_s-\theta_a)$, to which the nocturnal cooling approximates under different circumstances. Hence the effect increases with decrease of thermal capacity and conductivity and range of diurnal oscillation of temperature, but with increase of radiating power of the earth's surface, of diathermancy of the atmosphere for terrestrial radiation, and of altitude.

For reasons which have been given in the case of bodies near the earth's surface, there is little effect observed when the air is not perfectly quiet, since ventilation of the surface by air currents passing over it tends to reduce the earth's surface and all the lower strata of the atmosphere near it to the same temperature.

It must also be understood that in all which precedes with regard to nocturnal cooling, the temperature of the dew-point must be below the temperature to which the earth's surface cools. If not, as soon as this temperature is reached and condensation to dew or frost begins, the latent heat given out almost entirely arrests further cooling and introduces another condition which has not entered into the preceding equations and expressions.

131. The effects, as observed, mostly fall within the limit of 12°, as determined above from Pouillet's observations. The results of Mr. Wells, Mr. Daniell, and others who have made observations on nocturnal radiation show that a thermometer exposed on the ground during the night in an open space is cooled 6°, 7°, or even 8° below the air a few feet above. Wilson observed differences of temperature of nearly 9° between the temperature of the air and that of the surface of the snow. The great effect in this case was undoubtedly due mostly to the poor thermal conductivity of the snow, by which, we have seen, such effects are increased. Scoresly and Captain Parry have observed analogous depressions in the polar regions, where the temperature of the air was more than 20° F. below zero. The effect of a very low temperature would be in the direction of that of a diminished radiating power, as seen from (17), in which r and μ^{θ_a} are factors in the same denominator, but the effect would be small, since the value of μ^{θ_a} does not change much with any observed change of θ_a.

According to the observations of Mr. Glaisher,[67] "it appeared that at times when the sky was entirely covered with low cirro-stratus clouds, the readings of a thermometer placed on long grass was the same as in the air; that with the same clouds at a moderate elevation the reading of a thermometer in air has exceeded that on long grass by 3°; and on these clouds being high, this excess has amounted frequently to 5°; and if other than cirro-stratus covered the whole sky, this excess has been as large as 10°. At times when the sky has been free from clouds, but not bright, haze and vapor being present, the excess has amounted to 10°, 11°, or 12°; and at times when the sky has been both bright and clear, with the air calm, no mist, haze, vapor, or fog being present, this difference has frequently amounted to 14°, less frequently to 19°, and sometimes to 20°" (11° C.). This extreme difference still falls within the limit 12° C. given above.

These observations show the effect of different degrees of diathermancy of the atmosphere, or, in different words, of completeness of inclosure, since when the sky was entirely overcast with low clouds, the effect was nothing, but it gradually increased with increasing clearness of the atmosphere, until an effect was observed greater than the maximum effect observed by Wells or Wilson.

In one case the reading of a thermometer on raw cotton wool on long grass was 25° F. less, whilst another placed at 8 feet from the ground and fully exposed to the sky was 3.5° greater than that in air at the height of 4 feet. Thus a difference of 28.5° (16° C.) was observed in this case, the thermometer suspended in air, however, being above the standard height of 4 feet for such observations. Allowing several degrees for the effect of this greater height, this still exceeds the limit of 12° C. This limit, however, is the average for a number of ordinary clear nights, so that it must be expected that with the greatly varying diathermancy of the atmosphere for terrestrial radiations this limit will be surpassed on some nights when the atmosphere is exceptionally diathermanous.

Glaisher determined the relative radiating powers of the following different substances, that of long grass being 1,000:

Hare skin	1,316	Blackened tin	770
Raw white wool	1,222	White tin	657
Raw silk	1,107	Zinc	681
Unwrought white cotton wool	1,085	Iron	642
Long grass	1,000	Saw dust	610
Lamp-black powder	961	Slate	573
Glass	864	Garden mold	472
Copper	839	River sand	454
Lead	757	Stone	390
Jet black lamb's wool	741	Gravel	288

The metals in this list were placed on long grass. The observed effects must have depended very much upon the manner of exposure, and, for reasons already given, are only very rude relative measures of radiation. The greater observed effects, however, probably indicate, in general, substances of greater radiating powers.

132. The observed effect of nocturnal cooling is very much greater at elevated interior dry stations at some distance from the ocean than at coast stations. According to Blanford,[68] at the inland stations of Lahore and Nagpur, elevations 732 and 735 feet respectively, the difference between the minimum thermometer, 4 feet high, and the nocturnal radiation thermometer, close over short grass, or on a woolen pad, sometimes amounts to 15°, and in some extreme cases, even 18°, (10° C), while at the lower maritime stations of Calcutta, Bombay, and Madras it is only about one-third as much. This is undoubtedly due to the two causes of greater elevation and greater dryness at the former than at the latter stations. Small differences of elevation with the same diathermancy of atmosphere should produce a considerable effect. We have seen, § 90, that without any atmosphere the thermometer suspended above the earth's surface would stand more than 90° C below the temperature of the earth's surface. If we suppose that only about one-tenth of the heat of terrestrial radiation escapes into space, the effect is reduced to 12°, or to the average mean sky temperature for ordinary clear

nights. If nine-tenths of the heat radiated by the earth's surface at sea-level are radiated and reflected back by the whole atmosphere, then if this were proportional to the amount of atmosphere passed through, and it is approximately with the same diathermancy at all altitudes, then at an altitude of 1,000 feet, $\frac{1}{26} \times \frac{9}{10} = \frac{1}{30}$ nearly, more should escape, and if one-tenth cause, at sea-level, an effect of 10° C, as is observed in some cases, then at an altitude of 1,000 feet it should be 3.3° (6° F.) greater. At a considerably greater altitude this would be very much increased, being approximately in proportion to the altitude. But the principal part of the difference between the inland elevated stations and the maritime stations, as observed by Blanford, must have been due to the greater dryness of the atmosphere at the former stations, and this seems to indicate that aqueous vapor must have a great effect upon the diathermancy of the atmosphere, at least for terrestrial radiations.

That there is a great difference in the diathermancy of the atmosphere on different, apparently equally clear, nights, due probably to the difference in the amount of aqueous vapor, seems to follow from experiments made by Dr. Tyndall.[69] He found a difference between a thermometer suspended 4 feet above the ground and one on wool at the surface, November 11, 1881, 9.15 p. m., 11° F. The next evening, sky overcast, only a few stars visible, difference 4°. On November 10, snow a foot deep, wind very light from northeast, the following observations were made:

At 8.10 a. m., air 29°, wool 16°.

At 8.15, air 29°, wool 12°.

Up to this time the sun was invisible on account of cloud. At 8.50, air 29°, wool 11°. In this observation the sun shone a little on the air thermometer. At other times, however, when the air was serene and almost a dead calm, and the sky without a cloud, a difference only of from 4° to 6° was observed. Differences of this sort in the state of the atmosphere are frequently perceived without instrumental observation. At one time, with the atmosphere calm and very clear, or at least cloudless and all the stars shining, the air seems warm during the whole night, and the earth's surface does not cool down much by radiation into space. At another time, soon after sunset, with apparently only the same degree of clearness of the atmosphere, the temperature of everything falls very rapidly. There seems to be a great amount of heat radiated through the atmosphere into space. At such a time, if the air is calm, a man's shoulders, if he is in open space in the air, soon feel cold from the effects of the radiation, and he feels the need of a cape.

133. The effect of nocturnal cooling is often applied to a practical purpose in the manufacture of ice in the East Indies. "The natives of Bengal at the town of Hooghly, near Calcutta, make ice in fields freely exposed to the sky and formed of a black loam soil upon a substratum of sand. Shallow excavations are made in the soil, upon the bottoms of which are spread small sheaves of rice straw, and upon the top of this

light loose straw to the depth of a foot and a half, leaving its surface half a foot beneath the level surface of the ground. Upon this are exposed numerous shallow and very porous earthen dishes with water in them during clear nights. The ice is produced in large quantities when the temperature of the air is 16° to 20° F. above the freezing point."[70] The ice is formed mostly when the wind is from the northwest during the day and then subsides during the night.

The black loam, sand, and thick layer of straw, under the dishes of water, all have little thermal conductivity, and the straw at least very little thermal capacity per unit of volume, both of which we have seen, § 130, are necessary to obtain the greatest effect. The necessary dryness accompanies the northwest wind during the day, for the dew point, as we have seen, must be below freezing, and without the calmness at night there could be no nocturnal cooling. With a temperature much more than 20° F. above freezing, ice could not be formed, since then the effect of nocturnal cooling would have to exceed the limit of about 12° C.

Extreme temperatures.

134. In the oscillations of temperature, annual, diurnal, and abnormal, the greatest extremes are produced where the maxima or minima of all occur at the same time. Hence the highest temperatures at any one place, in general, are observed in July or August in the northern hemisphere soon after noon. The abnormal variations of temperature of course cause these to differ very much at different times within a few days. The greatest annual range of temperature we have seen, § 120, is in high latitudes in the interior of the continents. This causes very low winter temperatures on account of the low mean annual temperature of these latitudes, and on account of the large amplitude of annual inequality the mean summer temperature is not very much below that of the torrid zone. For instance, for Northeastern Asia, latitude 60° N. and longitude 130° E., we get from the tables of § 116 and 119 the mean annual temperature —9°, and from the table of § 120 the amplitude of annual inequality 27°. These give —36° for the mean of January and 18° for July. In the same manner we obtain on the polar circle for the same longitude —40° for the mean temperature of January. These, however, are merely approximate results for that region in general, but there are local causes which in places reduce these mean temperatures for January still much lower. It was formerly thought that Yakutsk has the coldest winter temperature in any explored region of the globe. According to Woeikoff,[71] however, this is not the case. He says: "Up to the present time Yakutsk, in Northern Siberia, has often been considered the place where the winters are coldest, while the minima observed during Arctic expeditions are believed to be the lowest known. Neither is true. The temperatures of Werkhojansk are the lowest of all." The following are the mean temperatures, in Fahrenheit degrees, for the year, January, and July:

Place.	Latitude.	Longitude.	January.	July.	Year.
	°	°	°	°	°
Serdze Kamen	67½ N.	173 E.	—13.1	…………	………
Ustjansk (two years' observation)	71 N.	136 E.	—38.9	52.7	2.8
Werkhojansk (one year)	67½ N.	134 E.	—55.2	60.1	4.8
Yakutsk (twenty-four years)	62 N.	130 E.	—41.4	63.6	12.4
Flooburg (one year)	83.5 N.	…………	—33.0	38.3	—3.5
Discovery Bay (one year)	81.5 N.	…………	—40.7	37.2	—4.2

He gives the following rational explanation of these extremely low temperatures. "During calms in clear nights (in high latitudes) the valleys are colder than the surrounding hills and slopes, because the cold air sinks down into the valleys. This continues during the day in high latitudes where the sun does rise, or at least has little power at midday. Even in middle latitudes, where the calms and clear weather prevail, the valleys are colder than the hills. The exceedingly low temperature at Werkhojansk is probably not common to the whole surrounding country."

The mean temperature of January for Werkhojansk, —55.2° (—48° C.), is considerably lower than it is at Yakutsk, and than —40°, as obtained from the preceding tables for the latitude and longitude nearly of Werkhojansk, and therefore does not seem to be common to the whole surrounding country, as Woeikoff supposed.

In these extreme cases the valleys and low grounds are cooled down, not only by their own nocturnal radiation, but likewise by a concentration of the effects of nocturnal cooling on the surrounding hills and slopes. The unusually increased depth of the cold air strata reduces very much the temperature of the earth's surface beneath.

The diurnal inequality in winter being extremely small in these high latitudes, there is little difference between the mean temperatures of night and day, but the abnormal variations are large, and when the minima of these occur during January we have the extreme low temperature of individual observations. On the Alert and the Discovery of the English North Pole Expedition there was observed in the beginning of March, during a long-continued cold spell, a minimum of —73.7° (—58.7° C.) on the former, and —59.6° on the latter. Newery saw in Yakutsk, January 21, 1838, the thermometer at —60° C. (—76° F.). At Jennesseisk (58° N., 92° E.), 39^{m} above sea-level, —58.6° C. was observed on the 12th of January, 1872.

In the interior of North America, on the same latitudes, the mean annual temperature is about the same, but the amplitudes of annual inequality, § 120, are 7° or 8° less, and hence the extreme winter temperatures are not so low, but at some places at and within the polar circle the mean temperature of the coldest months is as low as —40° C. With the effect of abnormal variations individual observations are often very much lower.

135. On lower latitudes, both on account of the higher mean annual temperature and smaller amplitudes of the annual inequality required

by them and given by observation, as seen from the tables of §§ 116, 120, the mean winter temperatures are very much higher, but on account of the greater diurnal oscillations here the minima of the nights may descend quite low, especially at times when they are affected by the minimum of an abnormal variation.

In the southern hemisphere on the same latitudes the annual inequality is greater than in the northern, all other circumstances being the same, as may be seen from the coefficients of the annual inequality of intensity of solar radiation, C, in the table of § 62. For instance on the parallel of 30° its value for the northern hemisphere is .0909, while for the southern hemisphere it is .1084, about one-fifth greater. Since all other circumstances may be assumed to be the same, the amplitudes of temperature oscillation must be in proportion, as is seen from (21), § 67 in which K_1 corresponds with and becomes C_1 at the top of the atmosphere, and the relation between the value of K_1 for the two hemispheres is the same as that between the values of C_1. On the parallel of 30° in the northern hemisphere, from Arabia to India the amplitudes of the annual temperature inequality are from 10° to 12°. (Table § 120.) Under similar circumstances, therefore, the amplitudes of the southern hemisphere on the same parallel should be $\frac{1}{5}$, or more than 2° greater, and consequently the maximum, or January, temperature greater by about the same amount. This in some measure accounts for the very high summer temperatures observed in the Argentine Republic, and in Australia, on and near that latitude, and where the summer temperatures are thought to be greater than in any other part of the world.

The value of C_1 at the equator being only .0101, or about $\frac{1}{9}$ of that on the parallel of 30° of the northern hemisphere, it should give rise, under the same circumstances—that is, in the interior of a continent with a dry soil and atmosphere—to a temperature inequality with an amplitude of only a little more than 1°, but on or near the ocean, where the inequalities of temperature, for reasons given in § 121 are comparatively small, it would be extremely small. The difference, therefore, between the mean temperatures of January and July at the equator is generally very small, but in the great Sahara desert it may amount to 4°, or the amplitude to 2°. Although the annual inequality at the equator, and within the tropics generally, is very small, yet, on account of the large diurnal inequality here, there is a considerable range between the two extremes of temperature at all inland dry stations where the range of temperature is not affected by oceanic influence. On account, however, of the high mean annual and diurnal temperature, we cannot have very low temperatures, but the maximum afternoon temperature of sandy dry soils, with little thermal conductivity, and so favorable for large temperature ranges, § 121, often becomes enormous, and on this account the temperature of the air becomes indirectly very great, though, for reasons which will follow, it is very much less than that of the soil.

Sir John Herschell observed at the Cape of Good Hope a soil temperature of 159° F. The soil temperature, also, on the Loango coast near the equator, has been observed to be, in numerous cases, 75°, often reach 80°, and once 84.6° C.[69]

In Senegambia and Sierra Leone the differences between the extremes of air temperature of day and night are very great. At St. Louis they are 7.9° as absolute minimum and 44.8° as absolute maximum. The southern coast of the Red Sea is the warmest region of the globe. The extreme temperature there is said to reach as high as 54° or 56° (133° F.).[65]

Relations between the temperature of the atmosphere and earth's surface.

136. We have seen that both the mean temperature of a body exposed to solar radiation and the amplitudes of the oscillations of temperature depend upon the relation between a', the absorbing power of the body for solar radiations, and r, the radiating power of the body. For the earth's surface the value of a' and r are very nearly or quite equal, but for the atmosphere we know that the former is less than the latter, and therefore the tendency is for the atmosphere to assume a lower mean temperature than the earth's surface, if it receives the same amount of solar heat, considering each one separately, and disregarding the interchange of radiations from the one to the other. In a state of static equilibrium of temperature where $a' < r$ we have seen that the temperature of the strata gradually decreases with increase of altitude, considering only the relations between radiation and absorption of the different strata of the atmosphere and the earth's surface with regard to their own heat and that received from the sun. Between the lower stratum of the atmosphere and the earth's surface there could not be any abrupt change or difference from the conditions of radiation and absorption alone if both received the same amount of heat from the sun, though the temperature of the former, on account of a' being less than r, would be a little less.

On account of the uncertainties in the relations between the absorbing and radiating powers, we do not know how much the mean temperature of the whole atmosphere would be below that of the earth's surface, without contact or interchange of radiations, yet with these there cannot be much between the lower stratum and the earth's surface. But the gradual decrease of temperature with increase of altitude, on account of a' being less than r, would still leave the mean temperature of the whole atmosphere less than that of the earth's surface.

137. Whatever the tendency may be to cause temperature of the atmosphere to be less than that of the earth's surface, this is greater in polar than in equatorial regions, and if the relations between radiation and absorption are such as to cause the surface to be warmer than the air in the equatorial, it may be the reverse in the polar regions. In Fig. 8, § 101, the difference between the heating effect of the sun's rays on a portion of air a near the equator and of one a' near the pole arises

in part from the difference in the normal intensities after having passed through different distances of the atmosphere. But the difference in the heating effect depends not only on this same difference in normal intensity at the earth's surface, but likewise upon the difference between the normal and vertical intensities for the two latitudes, which for the mean effect is very great, since in the polar regions the mean zenith distance, for all times that the sun shines on the place, is much greater, and therefore the mean solar vertical intensity much less, than in the equatorial regions. The temperature of the atmosphere, therefore, depending upon the normal intensities, and that of the earth's surface upon the vertical intensities, the tendency is for the temperature of the earth's surface to decrease from the equator to the pole with increase of latitude more rapidly than that of the lower stratum of the atmosphere, just as the latter does with reference to the upper strata of the atmosphere, § 101, on account of the greater difference in normal intensities of solar radiation.

From this weakening of vertical intensity in high latitudes results the enormous difference between the temperatures of the equatorial and polar regions, which would exist if there was no heat transferred from the former to the latter by means of ocean currents. And in this case there would be a great tendency for the polar regions to assume a temperature much lower than that of the atmosphere, but from contact the temperature of the lower strata would tend to the same, and this would decrease very much the rate of decrease of temperature with increase of altitude, if it would not reverse it.

With the great amount of heat transferred from equatorial to polar regions we cannot infer that there is a tendency in the earth's surface to assume a higher temperature than the air in the equatorial, or a lower temperature in the polar, regions. Comparisons of mean air with surface temperatures in the higher latitudes of Europe and Asia do not indicate certainly that there is on the average any difference, though there are considerable local variations.[68] From observations in India, however, made at three widely distant stations, the temperature of the ground, according to Blanford,[69] is 4° or 5° F., as a general rule, in excess of that of the air. At Trevandrum also (lat. 8.5°) the air temperature is 2.6° C. less than that of the ground at the depth of 1 meter.

138. The varying relations between air and earth temperatures, due to varying relations between normal and vertical intensities, as explained above in the case of differences of latitude, is best seen in a comparison of air and earth temperatures during the course of the day in clear weather. At sunrise there may be little difference between the earth and air temperature, the former being a little less generally from the effect of nocturnal cooling, but at first, while the air is receiving heat from the sun in proportion to the normal intensity and the earth's surface in proportion to the vertical intensity, which, we have seen, is comparatively small, the temperature of the air increases much faster than that of the earth's surface; but about two hours after sunrise,

when the rays come through the atmosphere with less obliquity and the vertical intensity becomes more nearly equal to the normal, the reverse takes place, and the earth's surface then begins to heat up faster than the air, until at noon and soon after the temperature of the former is much greater than that of the latter. In the afternoon the reverse takes place, so that one or two hours before sunset the more rapid cooling of the earth's surface has reduced its temperature below that of the air. The relations in this case are very nearly the same as those between the successive temperatures of a sphere suspended above the ground and those of a thin horizontal disk, of which an example has been given in § 94.

Some very interesting observations have been made on this subject by Hamberg.[72] He found that the temperature of the lower strata of the atmosphere during the night very near the surface increased from the surface upward, and continued to do so until about two hours after sunrise, after which a reversal took place, and there was the usual arrangement of decrease of temperature with increase of altitude. He therefore infers that "the elevation of temperature in the lower strata of the air is not in consequence of the heating of the soil, but rather of absorption by the air, or more accurately by the aqueous vapor in the air, of the heat which passes through the strata and which is reflected by the soil."

In this case there is no heat received by means of ocean currents when the sun shines obliquely upon the earth's surface, or lost when it shines more vertically, to equalize the temperatures, as in the case of differences of latitude, and we have somewhat the relations between air and earth temperatures which there would be in different latitudes if there were no ocean currents.

139. In the annual and diurnal temperature oscillations of both the air and the earth's surface the amplitudes are expressed by (64), in which the value of K' is different in the two cases; in the case of air it arises from the development of the normal intensity of radiation in a function of the time, and in that of the earth's surface from a development in a similar function of the vertical intensity. The relations between a' and r in the two cases are also different, a' being much less than that of r for the atmosphere. Therefore the amplitude of the temperature oscillations, both annual and diurnal, are greater for the earth than for the atmosphere. On account, however, of the length of period and the slowness of the change, there is little difference in the mean earth and air temperatures in January and July, since the lower strata of the air by continual agitation are cooled in winter and heated in summer so as to differ very little from the mean temperatures of the earth with which the air is in contact. The effect of the earth's temperature, however, on that of the air above is not so great as it is below, so that this causes the amplitudes in the oscillations of air temperatures, though less than those of the earth's surface, to be still greater

than those of the air above. The effect of this, it is readily seen, is to cause the temperatures in winter and during the night to approximate more nearly to the temperatures above, and hence to diminish the rate of decrease of temperature with increase of altitude at these times. But during the summer and the warmest part of the day, the effect is the reverse; it causes the temperatures below to differ still more from the temperatures above, and hence to *increase* the rate of diminution of temperature with increase of altitude.

In the diurnal oscillations the rate near the surface at night from the effect of nocturnal cooling is reversed for some distance above the earth's surface, the temperature being greater above than at the surface. As the earth cools the air in contact also cools when the air is calm, until the surface, and likewise the lower air strata are cooled very low, and the law of decrease of temperature is reversed. It is different during the day. The increase of the temperature of the earth's surface and of the lower strata in contact brings about a state of unstable equilibrium, § 38, from which at once arises a vertical interchange of air, by means of ascending and descending currents, which tends to equalize, in some measure, the temperatures above and below, so that although the earth's surface may be heated very much above the air immediately above, the decrease of temperature with increase of altitude never becomes very much greater than that of about 1° for 100 meters, corresponding to the initial state of unstable equilibrium. The effect of the heat of the earth's surface cannot be confined to the lower strata merely, as that of the cooling of the surface is, but as soon as the first stratum in contact with the earth is heated the effect is carried to those above.

140. The following table gives the results obtained by Glaisher from observations made during the period 1867–'70, at the height of 22 feet and 50 feet, and compared with those at the height of 4 feet. The difference between the readings of the upper and lower stations was taken and the plus sign was affixed to the difference when the temperature at the higher station was the greater, and the minus sign when it was the less.

Months.	22 feet and 4 feet.				Months.	50 feet and 4 feet.			
	9 a. m.	Noon.	3 p. m.	9 p. m.		9 a. m.	Noon.	3 p. m.	9 p. m.
1867-'70.					1869.				
January	+0.5	+0.2	+0.4	+0.6	October	+0.2	-0.5	+.07	+1.5
February	+0.2	0.0	+0.4	+0.5	November	+0.6	+0.5	+0.8	+1.4
March	-0.3	-0.2	0.0	+0.4	December	+0.9	+0.3	+0.5	+0.5
April	-0.6	-0.5	+0.2	+0.5	1870.				
May	-0.6	-0.4	-0.4	+0.5	January	+1.1	+0.3	+0.3	+0.9
June	-0.8	-0.9	-0.6	+0.8	February	+0.1	-0.3	+0.3	+0.5
July	-0.8	-0.8	-0.8	+0.7	March	-0.3	-1.8	-0.7	+0.7
August	-1.0	-0.5	-0.1	+0.9	April	-0.9	-2.2	-1.7	+1.4
September	-1.0	-0.6	0.0	+0.7	May	-2.4	-3.6	-2.8	+1.1
October	-2.2	-0.1	+0.6	+1.0	June	-2.4	-3.8	-3.1	+1.1
November	+0.2	+0.1	+0.6	+0.8	July	-1.8	-2.9	-2.8	+1.1
December	+0.5	+0.3	+0.4	+0.4	August	-1.7	-2.7	-2.0	+1.7

From these results it is seen that from 9 a. m. to 3 p. m. the temperatures during the summer were less at the upper station than at 4 feet, and during the winter the reverse. At noon, however, this was the case through the greater part of the year, while at 9 p. m. the reverse was the case, the temperature being greater above than below through the whole year, and this undoubtedly continued so through the whole night. In the winter months it is seen that the temperature was greater above than below for all times of the day and night, the effect due to the annual inequality of temperature oscillation entirely counteracting the reversing effect due to the diurnal inequality.

The following table contains the difference of temperature observed at the observatory of Montsouris between a thermometer 2^m and one 20^m above the earth's surface. The plus sign indicates a greater temperature below:

Hour.	March.	April.	May.	June.	July.	August.	September.
	°	°	°	°	°	°	°
1	—0. 18	—0. 72	−0. 44	—0. 28	−0. 79	—0. 68	—0. 98
2	0. 35	0. 71	0. 63	0. 36	0. 82	0. 75	1. 22
3	0. 27	0. 68	0. 78	0. 39	0. 75	0. 84	1. 25
4	0. 32	0. 43	0. 81	0. 37	0. 62	0. 82	1. 04
5	—0. 11	0. 38	0. 68	0. 24	0. 43	0. 81	0. 66
6	+0. 02	0. 50	0. 42	—0. 03	0. 16	0. 66	—0. 26
7	0. 13	0. 36	−0. 08	+0. 22	+0. 07	0. 39	+0. 02
8	0. 20	—0. 17	+0. 23	0. 47	0. 32	—0. 07	0. 18
9	0. 23	+0. 05	0. 52	0. 68	0. 49	+0. 37	0. 25
10	0. 37	0. 32	0. 85	0. 80	0. 60	0. 62	0. 34
11	0. 31	0. 58	1. 10	0. 88	0. 67	0. 92	0. 56
12	0. 36	0. 76	1. 00	0. 88	0. 64	0. 99	0. 86
13	0. 41	0. 83	1. 00	0. 82	0. 59	0. 90	1. 12
14	0. 49	0. 73	0. 99	0. 72	0. 48	0. 67	1. 22
15	0. 31	0. 55	0. 82	0. 53	0. 27	0. 37	1. 01
16	+0. 16	+0. 27	+0. 53	0. 33	+0. 04	+0. 01	+0. 52
17	−0. 03	−0. 04	—0. 02	+0. 06	−0. 21	—0. 35	—0. 08
18	0. 14	0. 31	0. 14	—0. 16	0. 41	0. 46	0. 62
19	0. 21	0. 51	0. 32	0. 18	0. 57	0. 59	0. 88
20	0. 20	0. 63	0. 47	0. 39	0. 66	0. 66	0. 90
21	0. 13	0. 69	0. 44	0. 37	0. 70	0. 67	0. 73
22	0. 06	0. 72	0. 35	0. 26	0. 70	0. 65	0. 54
23	0. 05	0. 72	0. 28	0. 26	0. 71	0. 62	0. 54
24	—0. 09	—0. 72	0. 32	—0. 24	0. 75	—0. 63	—0. 70

The following table of monthly rates of decrease of temperature with increase of altitude, in degrees centigrade per 100 meters, is taken mostly from the table given by Wild in his Temperatur-Verhältnisse, and from Meteorological Researches, Part III[73].

Month.	Kaukasus.	The Alps and Germany.	Hong-Kong.	Ceylon.	Schafberg.	Switzerland.	Geneva and St. Bernard.	Sacramento and Summit.	Pike's Peak and Colorado Springs.	Mount Washington, Burlington, and Portland.	Vera Crus and City of Mexico.
January	0.356	0.403	0.540	0.575	0.225	0.298	0.46	0.47	0.52	0.47	0.33
February	.428	.491	.560	.580	.330	.527	.53	.56	.64	.54	.36
March	.478	.601	.630	.575	.541	.674	.62	.61	.67	.53	.36
April	.558	.677	.760	.570	.590	.624	.63	.67	.70	.62	.31
May	.578	.695	.900	.590	.600	.710	.64	.55	.77	.63	.41
June	.606	.676	.900	.610	.600	.748	.66	.42	.76	.57	.51
July	.594	.652	.970	.620	.665	.702	.62	.33	.68	.61	.51
August	.600	.633	.860	.620	.540	.655	.60	.32	.68	.60	.54
September ...	.529	.599	.720	.605	.510	.571	.56	.36	.60	.58	.56
October	.462	.532	.610	.590	.370	.585	.55	.36	.62	.56	.52
November ...	.205	.445	.590	.580	.280	.518	.53	.43	.56	.55	.50
December ...	0.249	0.388	0.530	0.570	0.305	0.300	0.44	0.51	0.52	0.50	0.45
Year	0.470	0.566	0.720	0.590	0.455	0.576	0.57	0.466	0.64	0.563	0.45

It is seen from this table that in all parts of the earth the rate of decrease of temperature with increase of altitude is greatest in spring or summer and least in the early part of winter.

CHAPTER III.

THE GENERAL MOTIONS AND PRESSURE OF THE ATMOSPHERE.

I.—INTRODUCTION.

141. The motions of the atmosphere depend almost entirely upon differences of temperature between different places on the earth's surface; for with a uniform temperature the small effects depending upon a difference of tension of the aqueous vapor in the atmosphere would sensibly vanish. By the general motions and pressure of the atmosphere are meant here those motions and the resulting pressure which arise from the normal difference of mean temperature between the equatorial and polar regions of the globe and its annual variation, disregarding the more local variations which cause differences of temperature at the same time between different places on the same parallel of latitude, and all kinds of abnormal temporary disturbances of temperature. The general motions and pressure of the atmosphere, therefore, with their annual variations, embrace one general system extending over the whole surface of the earth.

The permanent cloud and rain belts, with their annual variations, and the dry zones of the earth, so far as they arise from the general motions of the atmosphere, must be regarded as parts of the same system, and they can, therefore, be more conveniently treated in the same connection.

In a merely descriptive treatise of the winds and of atmospheric pressure with their variations, and of evaporation and precipitation, of course each of these can be treated separately, but where an explanation of these and of their relations to one another is attempted, any one is so dependent upon all the rest that it is not possible to treat them separately.

II.—THE FUNDAMENTAL EQUATIONS.

First, in case of no rotation of the earth on its axis.

142. Let

U, V, X=Rectilinear co-ordinates in the directions respectively of south, east, and towards the zenith;

u, v, x=the corresponding velocities of relative motion in these directions respectively;

F_u, F_v, F_x=the forces in these directions respectively required to overcome the frictional resistances to each unit of mass;

p=the pressure of the atmosphere on a unit of surface;

ρ=its density.

We then have the well-known equations of absolute motion upon the earth at rest and with the co-ordinates U, V, and X fixed in space,

$$(1)\quad -\frac{dp}{\rho dU}=\frac{du}{dt}+F_u \qquad -\frac{dp}{\rho dV}=\frac{dv}{dt}+F_v \qquad -\frac{dp}{\rho dX}=\frac{dx}{dt}+F_x+g$$

in which dp in the several equations are the partial differentials with respect to U, V, and X respectively. The value of g, as here used, is to be taken without regard to signs in the direction of X.

The last of these equations is obtained by taking the differential of (5), § 8, with regard to m. This gives

$$dp=dm\left(g-\frac{du}{dt}-f\right)$$

But from (10), § 9, we get for unit of mass by putting $\sigma=1$, $dm=\rho dh$. With this we get

$$\frac{dp}{\rho dh}=g-\frac{du}{dt}-f$$

Changing h to X, u to x, and f to F_x to correspond with the notation above, and making g negative, since it is in a direction here contrary to the direction of motion X, we get (1_3).

From a similar expression of p arising from forces acting in the directions of U and V, we get the first two of (1) above. Since there is no component gravity in these directions, g vanishes in these equations.

The friction terms in these equations express the forces required to overcome resistances of all kinds to the motions of a unit of mass. The whole resistance to the horizontal motions for the stratum of air next to the earth's surface may be regarded as a function of the velocity relative to the earth's surface, and as acting in a direction contrary to that of motion. For strata above the earth's surface, it depends rather upon the differences of the relative velocities of contiguous strata. If the relative velocity of any stratum with regard to the strata immediately above and below are the same, then the stratum is acted upon by the one above it in one direction just as much as it is in the contrary direction by the stratum, and the stratum then cannot be said to suffer any resistance to its motion, and no part of the forces acting upon the stratum is required to overcome resistance to its motion. But if the relative velocities are such that the stratum is acted upon by the contiguous strata more in one direction than the other, then the stratum suffers resistance by the amount of the difference.

From (39) and (42′), chap. 1, we get as an approximate expression of the density in the case in which the atmosphere has an average amount of aqueous vapor in it,

$$(2)\qquad \rho=\rho'_0\frac{P}{P_0(1+.004\tau)}=\rho'_0\frac{p}{p_0(1+.004\tau)}$$

From (15), § 12, we likewise get, for a unit of surface, or $\sigma=1$,

$$dp=\Delta_0 G dP$$

With these two expressions (1) becomes in terms of the barometric pressure P.

$$(3)\qquad -\frac{d\log P}{\alpha dU}=\frac{du}{dt}+F_u \qquad -\frac{d\log P}{\alpha dV}=\frac{dv}{dt}+F_v \qquad -\frac{d\log P}{\alpha dX}=\frac{dx}{dt}+F_x+g$$

in which

$$(4)\qquad \alpha=\frac{\rho'_0}{P_0(1+.004\tau)}=\frac{1}{gl(1+.004\tau)}$$

since by (11), § 10, for the standard pressures p_0 and corresponding density ρ'_0, in which case h becomes l, we have

$$(4')\qquad \frac{p_0}{\rho'_0}=Gnl=gl$$

the last form of the expression being equal to the preceding by (7), § 9.

Since it is the logarithms of the pressure which comes into the expressions of (3), it makes no difference what kind of measure of the pressures is used, so that it may be either barometric or otherwise.

143. In the case of the earth's turning on its axis the directions of the co ordinates, U, V, and X, are not fixed in space, and the velocities u, v, and x become relative velocities with regard to these movable co-ordinates and the earth's surface, and are not absolute velocities. In this case there are certain terms arising from the earth's rotation which must be added to the first members above in order to make them applicable in the case of an atmosphere upon the earth with a rotation upon its axis.

Where there is a component of motion east or west, this motion, combined with that of the earth's rotation, gives rise to what is called a centrifugal force[*] in a direction perpendicular to the earth's axis, which, being resolved in the directions X and U, give rise to a term in each of these directions which depends upon the earth's rotation, and which would vanish if the earth were at rest. Let

r=the mean radius of the earth;
θ=angular distance from the north pole;
φ=angular distance in longitude east;
n=the gyratory velocity of rotation at unit distance;
ν=the gyratory or elative velocity east;
ρ=the perpendicular to the earth's axis.

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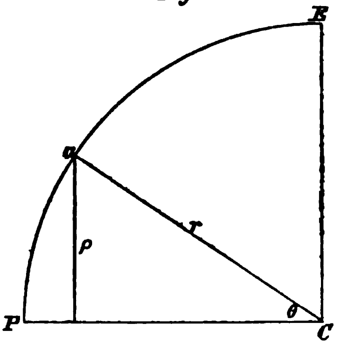
[*] It must not be understood that this force, so called, is a real force as defined in § 7, in which the velocity generated is in some given direction. The tendency of centrifugal force, where it is not restrained, is to increase the distance from the center, and when it is restrained from moving from the center, to cause pressure in the varying direction of the radius. This tendency arises simply from the inertia of the body by which it resists a change of direction.

We shall then have, neglecitng quantities of the order of the earth's eccentricity,

$$(5) \qquad \rho = r \sin\theta \qquad \nu = \frac{v}{\rho} = \frac{v}{r\sin\theta}$$

The centrifugal force for a unit of mass in the direction of ρ is $\rho\,(n+\nu)^2$, since $(n+\nu)$ is the whole gyratory velocity eastward at unit distance, due to both the earth's rotation, of which the gyratory velocity is n, and the relative velocity, v, of the air. This force being resolved in the direction of r and of the tangent at a toward the equator E, Fig. 1, perpendicular to r and parallel with the earth's surface, we get for the former,

Fig. 1.



$$(6) \qquad \rho(n+\nu)^2 \sin\theta = \rho n^2 \sin\theta + (2n+\nu)v \sin\theta;$$

and for the latter,

$$(7) \qquad \rho(n+\nu)^2 \cos\theta = \rho n^2 \cos\theta + (2n+\nu)v \cos\theta.$$

The first terms of the second forms of expression are independent of the relative velocity v, and consequently have nothing to do with the motions of the atmosphere relative to the earth's surface. The first of these, $rn^2 \sin\theta$, which becomes rn^2 at the equator, expresses the effect of the earth's rotation upon the earth's attraction, and being in the contrary direction—that is, from instead of toward the earth's center—diminishes a little the force of gravity toward the center. This, at the equator, amounts to $\frac{1}{289}$th of the whole. The second of these, $\rho n^2 \cos\theta$, in the direction of the tangent toward the equator, simply changes a little the direction of gravity arising from the earth's attraction alone. The resultant of the two is the whole force of gravity, usually denoted by g.

It is upon these forces that the elliptical figure of the earth depends, and it is the latter which would keep a body placed upon the elliptical surface, supposed to be entirely smooth and without friction, from sliding toward the pole. These forces, therefore, do not tend to produce any relative motions of the atmosphere upon the earth's surface, and consequently do not come into the equations of such motions.

The last term of the second member of (6) depends upon the eastward velocity v, and also tends to diminish gravity a little when v is positive, and the contrary when negative; but it is readily perceived that this term, with any ordinary value of v, is very small in comparison with the other, which, even at the equator, is a small part of gravity. In order to have them equal it would be necessary that ν should satisfy the following conditions:

$$n^2=(2n+\nu)\frac{v}{\rho}=(2n+\nu)\nu$$

which requires that $\nu=0.4n$ very nearly, and hence that the eastward relative angular velocity should be 0.4 of that of the earth's rotation. The last term, therefore, in (6), for any ordinary values of v, does not have any sensible effect upon gravity, and consequently may be neglected. But the last term in (7), though of the same order as the other, is an important term, since, being a horizontal force in the direction of the meridian, it must be added to the first member of (3_1), which expresses force in that direction, and although small in comparison with g in (3_3), yet it may be large in comparison with any force depending upon a gradient of pressure, or it may be sufficient to cause a considerable gradient of that sort.

Where there is a component of motion in the direction of the meridian, caused by any force acting in the plane of the meridian, there also arises a deflecting force in a direction at right angles to the meridian depending upon the earth's rotation. Since the force in this case, or at least a component of it, is either a centripetal or a centrifugal force with reference to the earth's axis, this deflecting force is deduced from the well-known principle of the preservation of areas in such cases. This principle, in this case, is expressed by

$$\rho^2(n+\nu)=r^2\sin^2\theta\,(n+\nu)=c \tag{8}$$

in which c is a constant depending upon the initial velocity v' at any polar distance θ', or distance ρ' from the axis of the earth's rotation. Differentiating and dividing by $r\sin\theta\,dt$, we get

$$2\cos\theta\,(n+\nu)\,r\frac{d\theta}{dt}+r\sin\theta\,\frac{d\nu}{dt}=0$$

From the differentiations of (5_2) regarding v and θ as variables, we get

$$d\nu=\frac{\sin\theta\,dv-v\cos\theta d\theta}{r\sin^2\theta}$$

We also have

$$r\frac{d\theta}{dt}=u \qquad \frac{v\cos\theta}{\sin\theta}=r\nu\cos\theta$$

By means of these last three equations the preceding one gives

$$-\cos\theta\,(2n+\nu)u=\frac{dv}{dt}$$

as a force depending upon the earth's rotation and the component of relative velocity u, acting in the direction of V and tending to accelerate the velocity v in that direction. Since the first member of (3_2) is a force in the same direction depending upon the gradient of pressure, this term must be added to this first member in order to have the equation expressing the condition to be satisfied, when the earth has a rotation on its axis.

With the terms $(2n+n)v \cos\theta$ and $-\cos\theta\,(2n+\nu)u$ added to the first members of the first two equations of (3), we get for the equations of the motions in the case in which the earth has a rotation on its axis,

$$
(9) \qquad
\begin{aligned}
-\frac{d \log P}{\alpha dU} &= -\cos\theta\,(2n+\nu)\,v+\frac{du}{dt}+F_u \\
-\frac{d \log P}{\alpha dV} &= \cos\theta\,(2n+\nu)\,u+\frac{dv}{dt}+F_v \\
-\frac{d \log P}{\alpha dX} &= g+\frac{dx}{dt}+F_x
\end{aligned}
$$

in which Napierian logarithms must be understood. By putting the first members of these equations equal 0, they become the equations of a projectile. In this case F_u, F_v, and F_x are the resistances to the motions in the several directions of U, V, and X by the air, and the amount of their resistances is equal to the momentum communicated to the air through which the body passes. In a vacuum of course these terms vanish.

The last one of (3), we have seen, is not sensibly affected by the earth's rotation.

The preceding method of obtaining the fundamental equations will perhaps be more readily understood by most readers, and has therefore been adopted here for the sake of greater simplicity, but a more elegant method will be found in Professional Papers of the Signal Service, No. VIII, and also in Meteorological Researches, Part I.

Besides these equations there is also another condition which must be satisfied, called *the condition of continuity*, by which is meant that in the motions of the atmosphere, just the same volume of air must flow out of any given space as flows into it, and vice versa, so that no two volumes shall occupy the same space, and no part of the space is left unoccupied.

The preceding conditions must be satisfied by the motions of the atmosphere in any case. In the case in which α or t in (4) does not vary in latitude or longitude, it is readily seen that all the conditions are satisfied by a state of rest. For we then have all the terms of the second members of the first two of (9) vanish and these equations are satisfied with P=a constant, and this constant is determined by the last of (9) with $x=0$, even where α, or t in (4), varies with change of altitude. In some cases, however, this state of rest may be an unstable equilibrium. Without motion of course the condition of continuity is satisfied.

In the general motions of the atmosphere the ascending and descending currents are so extremely slow that the forces required to overcome the inertia of the air in the case of accelerated velocities, expressed by Dx, and likewise the friction, expressed by Fx, are insensible and may be neglected. We therefore get from (9_3) for the vertical pressure, if we use h for X and suppose α to be independent of X, which by (4) requires τ to be the same at all altitudes,

$$(10) \qquad \log\frac{P'}{P}=\alpha g(h-h')$$

in which P' is the value of P where h becomes h'. Where α varies with h, this expression will be very nearly correct by using a mean value of α, that is, a mean temperature in (4). As both P and P' may vary with U and V, and likewise α by (4) where τ is different in different latitudes and longitudes, if we now regard P' as the value of P at the earth's surface where $h=0$, we get by taking the partial differentials with regard to U and V,

$$\frac{d\log P'}{\alpha dU}-\frac{d\log P}{\alpha dU}=gh\frac{d\log\alpha}{dU}$$
$$\frac{d\log P'}{\alpha dV}-\frac{d\log P}{\alpha dV}=gh\frac{d\log\alpha}{dV}$$

Subtracting these from the first two of (9) respectively, we get

$$(11) \qquad \begin{aligned} -\frac{d\log P'}{\alpha dU}&=-\cos\theta\,(2n+\nu)v+\frac{du}{dt}+F_u-gh\frac{d\log\alpha}{dU} \\ -\frac{d\log P'}{\alpha\, dV}&=\cos\theta\,(2n+\nu)\,u+\frac{dv}{dt}+F_v-gh\frac{d\log\alpha}{dV} \end{aligned}$$

These are the two equations of the horizontal motions, or components of motion, of the atmosphere, applicable where the vertical motions are so small that they may be neglected.

144. In the case of the general motions of the atmosphere, as defined in §141, α, by (4), becomes independent of longitude, and consequently of V. In this case, therefore, the last term of (11_2) vanishes and the equation then can be satisfied with $dP=0$. If the initial state is such that there is a gradient of pressure in the direction of V, this must soon disappear in the case of friction, since it could only tend to produce motion which would soon be destroyed by it. The first term of (11_2) then also vanishes.

The normal temperature of the earth at any time may be expressed in a function of the form:

$$(12) \qquad \tau=\Sigma A_s\cos s\theta,$$

in which $A_0=8.5°$ is the mean annual temperature on the parallel of 45°, and in the case of the mean annual temperature of the year, $A_2=-21°$ nearly, but A_1 is so small that it may be neglected. At special seasons of the year, however, the value of A_1 may be large.

From (12) we get, since $dU = r d\theta$,

$$\frac{d\tau}{dU} = \frac{d\tau}{r d\theta} = -\frac{1}{r} \Sigma s A_s \sin s\theta$$

and from (4)

$$\frac{d \log \alpha}{dU} = \frac{d\alpha}{\alpha dU} = -\frac{.004}{1+.004\tau} \frac{d\tau}{dU}$$

This, by means of the preceding, becomes

$$(12) \qquad \frac{d \log \alpha}{dU} = \frac{.004}{r(1+.004\tau)} \Sigma s A_s \sin s\theta$$

With this in the last term of (11_1), and putting the first member, and the last term of (11_2) equal 0, we get for the equations of the horizontal motions in the case of the general motions of the atmosphere,

$$(13) \qquad \begin{aligned} -\frac{d \log P'}{\alpha dU} &= -\cos\theta\ (2n+\nu)\ v + \frac{du}{dt} + F_u - \frac{.004gh}{r(1+.004\tau)} \Sigma s A_s \cos s\theta \\ 0 &= \cos\theta\ (2n+\nu)\ u + \frac{dv}{dt} + F_v \end{aligned}$$

To determine the motions of the atmosphere which satisfy these equations, and at the same time the condition of continuity, is a problem which cannot be completely solved, both on account of its complexity and also on account of the uncertainty in the laws and constants of friction in the motions of the atmosphere. All that can be done in the more general cases is to determine the general tendency of the temperature disturbances of static equilibrium and the approximate system of motions satisfying the conditions, leaving quantitative results, especially where they depend upon friction, to be determined from observation. If the problem could be completely solved for the most general case the proper order would be to take this at the start and to deduce the results for the special cases by way of corollary from the more general. As it is, it will be best to take the most simple cases first, since the results obtained from these will aid us in solving, so far as possible, the more general and complex ones.

III.—MOTIONS AND PRESSURE IN CASE OF NO FRICTION.

Deflecting force due to the earth's rotation.

145. The most simple case under this head is that of a body free to move in any horizontal direction without friction upon the earth's elliptical surface, supposed to be entirely smooth. In this case the terms in (13), depending upon pressure, friction, and temperature disappear, and we have left

$$(14) \qquad 0 = -\cos\theta(2n+\nu)v + \frac{du}{dt} \qquad\qquad 0 = \cos\theta(2n+\nu)u + \frac{dv}{dt}$$

If we confine ourselves to the case in which the relative velocities are so small that they can be neglected in comparison with the absolute motions depending upon the earth's rotation, then ν can be neglected and they become

$$(15) \qquad \frac{du}{dt}=2nv\cos\theta \qquad\qquad \frac{dv}{dt}=-2nu\cos\theta$$

By (1), § 7, the first members of these equations express forces for unit of mass in the directions of v and u respectively, and they are consequently the deflecting forces due to the second members, which depend upon the earth's rotary velocity n. If we now put

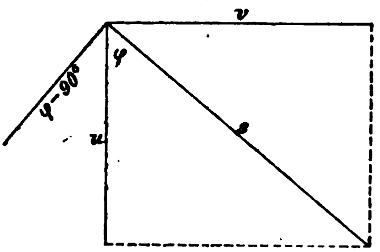
F =the resultant of these forces;
s =the resultant velocities of u and v;
φ =the angle between the directions of s and u;
φ'=the angle between direction of resultant force F and of u

we shall have by means of (14)

$$(16) \qquad \begin{aligned} F&=\sqrt{\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2}=2ns\cos\theta \\ \tan\varphi'&=\frac{dv}{dt}:\frac{du}{dt}=\frac{-u}{v}=\frac{-\cos\varphi}{\sin\varphi}=\frac{\sin(\varphi-90^\circ)}{\cos(\varphi-90^\circ)}=\tan(\varphi-90^\circ)\end{aligned}$$

Hence $\varphi'-\varphi=-90^\circ$, and consequently the resultant of the deflecting forces is at right angles to the right of the direction of s, the resultant velocity of motion. In the southern hemisphere the value of $\cos\theta$ is negative, and making v, instead of u, negative in the expression of $\tan\varphi'$, we get $\varphi'-\varphi=90^\circ$, and consequently the direction of the resultant force is then to the left of the direction of resultant motion. Of course, the forces in (15) vanish where there are no initial velocities u and v. Hence, *whenever a body moves in any direction on the earth's surface, there is a force*[*] *arising from the earth's rotation which tends to deflect it to the right in the northern hemisphere and the contrary in the southern.*

FIG. 2.



At the equator, where $\cos\theta=0$, this deflecting force, by (16_1) vanishes.

This force, divided by gravity, gives us the tangent of the angle by which a plumb-line would change its direction in consequence of such a force, and consequently the gradient, which is caused in the surface of a liquid in motion in consequence of this force. This tangent is expressed by

$$(17) \qquad \frac{F}{g}=\frac{2n}{g}s\cos\theta$$

The value of g must be expressed in the same measures as s. With the data in Appendix, Table XIV, we get

(18) $\frac{2n}{g}$=0.00001487, if s is expressed in meters per second,
=0.00000665, if s is expressed in miles per hour,
=0.00000454, if s is expressed in feet per second.

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[*] This force is of the same nature as centrifugal force and is not a real force,

Also $F : g$ is the ratio between the lateral pressure and the weight of a body constrained to move in a straight line in any direction on the earth's surface.

In the case of the atmosphere it is the gradient in angle of a stratum of equal pressure, or, in other words, it is the ratio between the horizontal distance at right angles to the direction of motion, and the perpendicular ascent of the incline.

Motion of a free body on the earth's surface.

146. By (16) a body moving in any direction is being continually deflected to the right or the left, according to hemisphere, with a force which is proportional to the velocity s. The radius of curvature of its path must be such that the centrifugal force is equal to the deflecting force. Putting

$\rho =$ the radius of curvature,
$m =$ the angular velocity of this radius;

the centrifugal force for unit mass is φm^2, and we therefore have

$$\rho m^2 = 2ns \cos \theta \qquad \rho m = s$$

These give

$$m = 2n \cos \theta \qquad \rho = \frac{s}{2n \cos \theta} \tag{19}$$

Where ρ is so small that θ may be regarded as a constant for all parts of the path described, it becomes sensibly a constant and the body may be regarded as moving in the circumference of a circle. Putting then τ for the period of revolution, we get

$$\tau = \frac{2\pi}{m} = \frac{2\pi}{2n \cos \theta} = \frac{1 \text{ day}}{2 \cos \theta} = \tfrac{1}{2} \text{ day} \times \sec \theta \tag{20}$$

Hence the period of revolution is independent of the initial velocity s with which the body is started.

If in (14) we put $\nu = -2n$, that is, give the body a westward relative velocity equal to twice the velocity of rotation, we get $D_t u = 0$, and with the initial velocity westward we have $u = 0$, and hence, by (14_2) $D_t v = 0$, that is, uniform velocity v. With such an initial velocity, therefore, the body would move around with uniform velocity on the same parallel of latitude and perform a revolution in a half day.

Examples.

1. If a river in the northern hemisphere on the parallel of 45° flows in any direction at the rate of 4 miles per hour, and is 1 mile in width, how much higher does the water stand at the right-hand than at the left-hand bank?

One mile is 63,360 inches, and cos 45°=.707. Hence, (17) and (18),

$$63360\frac{F}{g}=0.00000665\times 4\times .707\times 63360=1.2 \text{ inches.}$$

2. If a railroad train on the parallel of 45° runs at the rate of 40 miles per hour, what would be the lateral pressure per ton of the weight of the train on the side of the rails if both were on the same level?

In this example we get $F:g=0.000188$. Hence the lateral pressure is about 0.38 of a pound per ton of 2,000 pounds.

3. If the atmosphere moves at the rate of 30 miles per hour, what is the ascent of the gradient of equal pressure in the distance of one degree of a great circle?

4. If a free body without friction were to receive a velocity in any direction of 20 feet per second, on the parallel of 40°, what would be the radius of curvature, and also the period of revolution, approximately in the circumference of a circle?

From (19_2) and (20) we get in this example

$$\rho=\frac{20}{2\times .00007292 \cos 50^\circ}=213345 \text{ feet}=40.4 \text{ miles}$$

$$\tau=\frac{1 \text{ day}}{2 \cos 50^\circ}=1.2846 \text{ day}.$$

5. In example 1, what radius of a curvature must the river have so that the centrifugal force will exactly counteract the deflecting force of the earth's rotation and leave the surface level?

Motion and pressure of atmosphere in case of no temperature disturbance.

147. If a particle of air or any body were moved toward or from the pole by any force acting in the plane of the meridian, and there were no friction to impede its motion east or west, we should have in (13_2) $F_v=0$, and the equation would then give by integration equation (8) from which it was deduced by differentiation. This equation shows that the gyratory easterly velocity $(n+\nu)$, is inversely as ρ^2 and the first member of that equation expresses the double area described in a unit of time by the projection of the radius ρ on a plane perpendicular to the axis of rotation, the necessary equality of these areas, where there is no force or component of force acting perpendicularly to the plane of the meridian, being the principle from which this equation was deduced. The relations, then, between the absolute angular velocity eastward, $(n+\nu)$, and the radius vector ρ, are precisely the same as in the case of the heavenly bodies moving around the sun. As these approach or recede from the sun the angular velocities are increased or decreased, and if, as in the case of the comets, they approach very near the sun, the angular velocities become enormous. So

in the case of a body on the earth's surface, forced to move nearer to, or recede further from, the pole or axis of rotation by forces having no component perpendicular to the plane of the meridian, if the body is moved toward the pole and ρ becomes small, the value of $(n+\nu)$ becomes very large, and as it recedes toward the equator it becomes smaller but cannot vanish and change sign since the area described in a unit of time must remain constant.

The value of the constant c in (8) depends upon the initial state of the body before motion toward or from the pole takes place. If this is a state of rest and the body starts from the polar distance θ_0, then we get from (8)

$$c=r^2n\sin^2\theta_0$$

With this value of c (8) gives, since $r\nu\sin\theta=v$,

$$v=\left(\frac{\sin^2\theta_0}{\sin\theta}-\sin\theta\right)rn \tag{21}$$

It is seen from this expression that v, and consequently $\nu=v:r\sin\theta$, depend upon the polar distance θ_0 of the body before starting. It has usually been assumed in explanations of the east and west components of the atmosphere given in text-books that $\rho(n+\nu)$, the absolute easterly velocity, is a constant for all latitudes, and hence if it has a velocity of 1,000 miles per hour at the equator, it must have the same on any other parallel, however near the pole; and hence the relative easterly velocity v, as the body approaches the pole, approximates to that velocity but cannot exceed it, whereas, by (21), it becomes enormously great near the pole and approximates to infinity as the body approaches the pole.

148. If the initial state of the whole atmosphere were either that of rest or motion and an interchanging motion of the particles between the equatorial and polar regions should be caused by any forces acting in the planes of the meridians only, each particle would have an initial absolute linear velocity depending on the initial relative velocity and the latitude of the place before the interchanging motion commenced, and hence the value of the constant c in (8) would be different for different particles. If we now suppose that the different strata of air have a mutual action upon one another through friction, but that there is no friction between the atmosphere and the earth's surface, all the particles, whatever may have been their initial state or starting point, will be brought to have the same relative easterly velocity at all altitudes on the same parallel of latitude, and we must in this case have the sum of the projected areas for all the particles of the whole atmosphere equal to the sum of the initial areas before the interchanging motions commenced, since the equality of those areas cannot be affected by the mutual actions of the particles upon one another. The value of c in (8)

will then be the same for all the particles of the atmosphere, and putting C for this common value, we shall have, by putting m for the mass,

$$(22) \qquad C=\frac{\int c dm}{m}=\frac{\int r^2 \sin^2\theta(n+\nu')dm}{m}=\tfrac{2}{3}r^2(n+\nu'')$$

in which c is the initial value for each particle, and in which

$$\nu''=\frac{\int \sin^2\theta\nu'}{m}$$

ν' being the initial value of ν for each particle before the interchanging motion between the equatorial and polar regions commenced.

If we suppose $\nu'=0$ for each particle, that is, that the initial state was that of rest relative to the earth's surface, then ν'' in (22) vanishes and we have

$$C=\tfrac{2}{3}r^2n$$

With this value of C, which is now the common value of c for all the particles, (8) gives in case the initial state of the atmosphere was that of rest relative to the earth's surface,

$$(23) \qquad v=r\nu\sin\theta=\left(\frac{2}{3\sin\theta}-\sin\theta\right)rn$$

If in this equation we put $v=0$, we get

$$(24) \qquad \sin\theta=\sqrt{\tfrac{2}{3}}$$

to which corresponds $\theta=54° 44'$ for the polar distance, or 35° 16′ for the latitude, at which v vanishes and changes sign. Between this parallel and the pole it is positive, but between it and the equator, negative.

149. From (9) we get, in case of no friction, by putting for u and v and $D_t u$ and $D_t v$ their equals $D_t U$ and $D_t V$ and $D_t^2 U$ and $D_t^2 V$

$$-\frac{d\log P}{\alpha}=-\cos\theta(2n+\nu)\frac{dv}{dt}dU+\frac{d^2U}{dt^2}dU$$

$$-\frac{d\log P}{\alpha}=\cos\theta(2n+\nu)\frac{dU}{dt}dV+\frac{d^2V}{dt^2}dV$$

$$-\frac{d\log P}{\alpha}=gdX+\frac{d^2X}{dt^2}$$

in which it must be remembered that the differentials in the first members are the partial differentials with regard to U, V, and X, respectively.

Now we can write

$$\frac{dV}{dt}dU \text{ for } \frac{dU}{dt}dV,\ udu \text{ for } \frac{d^2U}{dt}dU, \text{and } vdv \text{ for } \frac{d^2V}{dt}dV$$

With the first of these changes the two terms depending upon $(2n+\nu)$ become identical with contrary signs, and hence disappear from the sum of the three equations. With the other changes in the equations we get, in common logarithms, by integrating from P_0 and regarding α as constant,

$$(25)\quad (\log P_0 - \log P) = g\alpha(X_0 - X) + \tfrac{1}{2}\alpha(s^2 - s_0^2)$$
$$= \frac{X_0 - X}{18401(1 + .004\tau)} + \frac{s^2 - s_0^2}{360940(1 + .004\tau)}$$

in which

$$(26)\qquad s^2 = u^2 + v^2 + x^2$$

s_0 being the value of s when u, v, and x become respectively u_0, v_0, and x_0.

The last form of the second member of (25) is obtained by means of (4) and Appendix, Table XIV.

As the preceding integrations have been made upon the hypothesis that α is constant, we must suppose that τ in (4) is a constant, or sensibly so. In order to take into account the variation of α, which by (4) is a function of τ, it would be necessary to have τ, and thus α, expressed in a function of X, U, and V. There would thus be terms introduced into the partial differential equations depending upon the variation of temperature, the integration of which would give the effect of the temperature variation in different parts of the atmosphere. These, however, would be very small in this case in comparison with the other terms depending upon the earth's rotation.

Where the initial and final point of integration are both in the same horizontal stratum, the first term of the second member of (25) vanishes, and we get

$$(27)\qquad (\log P_0 - \log P) = \tfrac{1}{2}\alpha(s^2 - s_0^2 = \frac{s^2 - s_0^2}{360940(1 + .004\tau)}$$

Where the interchanging motion between the equatorial and polar regions is infinitely small u and x in (26) vanish, and we then have

$$(27')\qquad (\log P_0 - \log P) = \frac{1}{2}\alpha(v^2 - v_0^2) = \frac{v^2 - v_0^2}{360940(1 + .004\ \tau)}$$

If we suppose the initial point of integration to be on the equator, where $\sin\theta = 1$, and where, consequently, by (23) $v_0 = -\tfrac{1}{3}rn$, by using this value and the general value of v in (23), the preceding equation becomes by (4) and Appendix, Table XIV,

$$(28)\qquad (\log P_0 - \log P) = \frac{1.1946}{1 + .004\tau}\left(\frac{2}{9\sin^2\theta} + \frac{1}{2}\sin^2\theta - \frac{13}{18}\right)$$

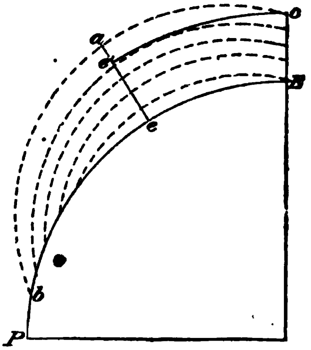
Differentiating this equation, and putting the differential of the first member equal 0, we get

$$\sin^2\theta = \frac{2}{3}$$

where the pressure is a maximum, the same as (24), and hence this maximum occurs where v vanishes and changes sign as the body or particle passes from the equatorial to the polar regions or the reverse, and which, we have seen, is on the parallel of 35° 16′. This, however, must be understood to be only in the case where the initial state of the atmosphere is that of rest, since the value of v here used, (23), is the value obtained upon that hypothesis.

Since sin θ becomes very small near, and vanishes at, the pole, $-\log P$ in (28) becomes very great near the pole and consequently P very small and vanishes at the pole where sin $\theta=0$. Hence, in this case of no friction between the atmosphere and the earth's surface, any stratum of equal pressure, however rare it may be, and however high it may be in the equatorial and middle latitudes, must be brought down to the earth's surface near the poles. Such strata, therefore, have a bulging up with maximum height on the parallel of 35° 16′, a considerable depression at the equator, and a great depression in the polar regions, the higher strata coming to the surface very near the poles, but lower strata at intervening latitudes between this and the parallel of the maximum, as represented in Fig. 3. This is shown by computation by means of (28) with any assumed value of P_0 at the equator and of P, however small. The value of θ obtained with these assumed values is the polar distance where the pressure is equal P.

Fig. 3.



Where the initial and final point of integration are both in the same stratum of equal pressure, we have $P_0=P$, and the first member of (25) vanishes. We then have

$$(29) \qquad s^2-s_0^2=2g(X-X_0)$$

which is the expression for the velocity of a body falling from X to X_0 where the initial velocity is s_0. In the case of an infinitely small interchanging motion between the equatorial and polar regions, since u_0 and x_0 then vanish, this becomes

$$(30) \qquad v^2-v_0^2=2g(X-X_0)$$

On the parallel of 35° 16′ we have $v_0=0$, and hence

$$(31) \qquad v=\sqrt{2g(X-X_0)}$$

The value of v for the point b on the earth's surface in the stratum of equal pressure *cab*, Fig. 3, is the same as the velocity acquired by a body in falling perpendicularly from a to e. Likewise the value of v at the equator, where the negative value must be used, is the velocity acquired by a body in falling from a to e' on the same level as c at the equator.

By whatever path a body comes from a higher to a lower level, provided the path does not form angles in changing its course, the velocity is the same at the lower level. The pressure of the sides of the channel being alway at right angles to the direction, it neither accelerates nor retards the velocity. The same is the case with the deflecting forces of the earth's rotation. They have no effect upon the final velocity at the lower level, but a very great one upon the final direction; and unless the initial values of u and x in (26), that is, u_0 and v_0, are very great, the final direction becomes nearly east or west.

In the preceding results the motions are due to an initial impulse giving the initial velocities u_0, v_0, and x_0, and not to any constant temperature disturbance, since α, which is the only quantity depending upon the temperature, has been treated as a constant.

Examples.

1. If a body at rest at the equator is moved without friction toward the pole, by a force acting only in the plane of the meridian, what is its relative eastward velocity at the parallel of 60°? ($\theta=30°$), (23), (Appendix, Table XIV,) ($\sin\theta_0=1$).

2. What, in meters per hour, on a parallel of 75°?

3. If the body at rest on the parallel of 60° is moved without friction to the equator, what is its westerly velocity there?

4. With the value of $P_0=0.76^m$ at the equator, and $\tau=0$, what by (28) would be the maximum pressure on the parallel of 35° 16′?

5. What, on the parallel of 70°, with $\tau=20°$? (28)

6. At what polar distance θ would the stratum of equal pressure of $P=0.2^m$ touch the earth's surface, the value of P_0 at the equator being 0.76^m?

7. What is the maximum height of a stratum of equal pressure at a, Fig. 3, which comes to the earth's surface at the polar distance $\theta=20°$? (23), (31), ($X_0=0$.)

8. What is the height at the equator?

Special solution in case of temperature disturbance.

150. Another solution of (13) in the case of no friction, but taking into account difference of temperature between the equator and poles, can be obtained by putting $u=0$, and $v=$a constant, in which case there is no interchanging motion between the equator and the poles; and this is a very important solution, inasmuch as it is approximately the real solution in the case of nature in which friction comes in. Putting $u=0$ and $v=$a constant (13_2) is satisfied in the case of no friction, and $D_t u$ and F_u vanish and (13_1) becomes

$$(31)\qquad \frac{d\log P'}{\alpha dU}=\cos\,\theta\;(2n+\nu)\;v+\frac{.004gh}{r\,(1+.004\tau)}\Sigma sA_s\sin s\theta$$

Putting v' for the value of v where $h=0$, we get, since the first member is constant with regard to altitude,

$$(32)\qquad \frac{d\log P'}{\alpha\,dU}=\cos\,\theta\;(2\,n+\nu)\;v'$$

From these two equations we get

$$(33) \qquad v-v'=-\frac{.004\,gh}{r\,(1+.004\,\tau)}\frac{\Sigma\,sA_s\sin S\theta}{\cos D(2n+\nu)}$$

In these expressions, as in (13), α and τ are supposed to be constant for the different strata, and where not, it is seen that this expression varies very little with a small change in the value of τ, and with the value of τ for either of the strata corresponding to h, or better with the mean of all, this expression becomes sensibly correct. Since ν is very small, also, in comparison with $2n$, it can be either entirely neglected, or the mean value of all the strata can be used, and in either case the error will be of no importance.

In the case of the mean temperature of the earth A_1 in (31) vanishes, or very nearly, §145, and we then have very nearly, using only the second term in (12),

$$(34) \qquad v-v'=-\frac{.016\,gh\,A_2\sin\theta}{r(2n+\nu)\,(1+.004\,\tau)}$$

Since A_2 (12) is negative where the temperature increases from the pole toward the equator, ($v=v,$) increases with increase of altitude and in proportion to the difference of altitude if we regard the small effects of ν and τ in (34) as being a constant for all altitudes. In the case of no friction, v' may have any assumed value according to the initial motions given to the atmosphere.

If $v'=0$, in (32) we get P'=a constant with reference to U or the polar distance, and the pressure is the same for all latitudes between the equator and the pole.

151. From (5_2) and (34), neglecting ν in comparison with $2n$ in the latter, we get, in case of the mean annual temperature of the earth, putting ν' for the value of ν at the surface,

$$(35) \qquad \nu'-\nu'=-\frac{.016\,ghA_2}{2\,r^2n\,(1+.004\,\tau)}$$

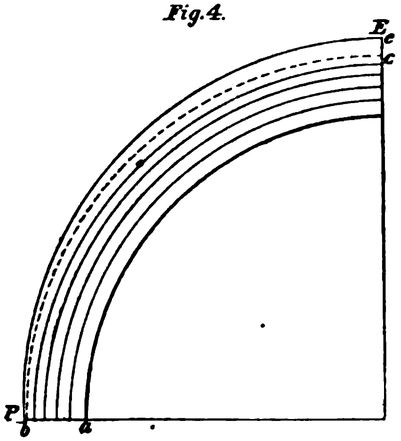
Hence the angular relative easterly velocity increases from the earth's surface in proportion to the increase of altitude h, and it is the same from the equator to the pole except so far as it is slightly affected by a variation of τ with change of latitude. The rate of increase at all latitudes is also in proportion to the temperature coefficient A_2, that is, the difference of temperature between the equator and the poles. When $\nu'=0$, the ratio between the increase of angular absolute rotation of a stratum of equal pressure of the atmosphere at the altitude h to that of the earth itself is

$$(36) \qquad \frac{\nu}{n}=-\frac{.008\,ghA_2}{r_2n_2\,(1+.004\,\tau)}=-2.317\frac{A_2}{1+.004\,\tau}\cdot\frac{h}{r}$$

Since A_2 is negative this makes the angular absolute velocity of rotation gradually increase from the surface of the earth, where in this case it is supposed to be 0, in proportion to the altitude h.

A popular explanation of this result is that the increasing temperature from the pole to the equator according to the law of the second inequality of (12), expands the atmosphere upward more at the equator *E*, Fig. 4, than at the pole *P*, and causes the strata of equal pressure to decline from the equator to the pole, and the more so the greater the altitude *h*. On this account the air tends to flow from the equator toward the poles in all the strata except the one next the earth's surface, where $h=0$, this tendency being in proportion to *h*. But in order to counteract this tendency and have the conditions satisfied without any motion from the equator towards the poles or the reverse, it is necessary for the strata above to have a greater angular absolute motion of rotation around the earth's axis than the earth's surface has, in order that the tangential component of the centrifugal force which tends to drive the air from the poles toward the equator may be greater above than it is at the earth's surface where it is just sufficient to sustain the spheroidal figure of the earth and the static equilibrium of the lower stratum of the atmosphere when at rest relatively to the earth's surface.

Fig. 4.



152. If in (10) we regard *h* and *h'* as being the heights of the strata of equal pressure of *P* and *P'* respectively, then *h* and *h'* become functions of *U* in the case we are now considering, and we get by differentiation

$$0=d(h-h')+(h-h')d\log\alpha$$

From this and (12') we get, using only the term depending upon A_2 here,

$$\frac{d(h-h')}{dU}=-(h-h')\frac{d\log\alpha}{dU}=-\frac{.008(h-h')A_2\sin 2\theta}{r(1+.004\tau)}$$

or putting for *U* its equal $rd\theta$

$$d\log(h-h')=-\frac{.008A_2\sin 2\theta}{(1+.004\tau)}d\theta$$

The integration of this gives approximately, regarding τ as a constant,

$$\log\frac{h-h'}{h_0-h'}=-\frac{.008A_2\sin^2\theta}{1+.004\tau} \qquad (37)$$

in which h_0 (*ab* in Fig. 4) is the value of *h* at the pole. In the case of no east or west motion of the atmosphere at the earth's surface, *P'* by (32) becomes a constant for all latitudes.

Wherever *v'* is positive—that is, where the motion of the atmosphere at the earth's surface is easterly there is by (32) a gradient of pressure increasing from the pole toward the equator. If the value of v' is positive in the higher latitudes and the contrary in the lower latitudes, then there is an ascending gradient from the pole up to where v' vanishes and changes sign, but a descending gradient from that latitude to the equator. Instead of the strata then being as in Fig. 4, there will be a bulging up in the middle latitudes of the strata of equal pressure, and the greatest pressure will be where $v'=0$.

Examples.

1. If the atmosphere is at rest at the earth's surface, what must be the relative increase of angular velocity of rotation of a stratum at the height of 3 miles for the state of mean annual temperature of the globe?

A mile being equal to 1609.3^m, we get by (36) and Appendix, Table XIV, putting $\tau=15^\circ$

$$\frac{\nu}{n}=\frac{2.317\times1609.3\times3\times21}{6367323(1+.004\times15^\circ)}=\frac{1}{29.5}$$

Hence the rate of angular rotation must be increased more than $\frac{1}{30}$ part.

2. What would be the absolute increase on the parallel of 40° at the height of 3 miles?

3. If at the pole $h_0=5{,}280$ feet or one mile and the difference of temperature between the equator and the pole is 40° ($A_2=-20^\circ$), what is the value of $(h-h_0)$ or oe in Fig. 4? (37), ($h'=0$)

IV.—MOTIONS AND PRESSURE IN CASE OF FRICTION.

For mean annual temperature of the earth.

153. If the initial relative velocities east or west are not such as to satisfy (33), then the disturbing force arising from a difference of temperature, expressed by the last term of (13_1), is not exactly counteracted by the deflecting forces, §145, and the residual uncounteracted part gives rise to an interchanging motion between the equator and the poles, and then u in (13_2) has a value. For instance, if the initial state of the atmosphere were that of rest, we should have $v=0$, $u=0$, and $d\log P'=0$, and then we should have

$$\frac{du}{dt}+F_u=\frac{.004gh}{r(1+.004\tau)}\Sigma sA_s\cos s\theta \tag{38}$$

In the case of the mean annual temperature of the earth, neglecting the variations of the seasons, (38) becomes

$$\frac{du}{dt}+F_u=\frac{.008ghA_2\cos 2\theta}{r(1+.004\tau)} \tag{39}$$

The value of A_2, §144, being negative in this case, it is seen that for all altitudes h, the tendency of the temperature disturbance is to produce an initial motion in the directions of the poles. The term F_u only acquires value after the initial motion. This initial motion of all the strata toward the poles only, increases the pressure toward the poles and diminishes it toward the equator and gives rise to a pressure gradient below, which causes a counter motion in the lower strata, and there is a stratum at a certain altitude in which there is no motion between the equator and the poles. The velocities in this interchanging motion and the altitude of the stratum of no motion must be such as to satisfy the condition of continuity, which requires that just as much air must flow from the poles below as flows toward them above. This condition is expressed by

$$\int u dm = 0 \tag{40}$$

for each vertical column of atmosphere, m being put for the mass of the air in the column.

If there was no friction and the initial values of v were not such as to satisfy (33), then the temperature disturbance would cause a continually accelerated interchange of air between the equator and the poles, but where there is friction the velocity is accelerated only until the friction term F_u (39) becomes equal to the disturbing force expressed by the last member, after which there is uniform motion, and consequently $D_t u$ of (39) vanishes.

Without any temperature disturbance the second member of (39) vanishes and it becomes

$$\frac{du}{dt} = -F_u \tag{41}$$

which indicates that any initial velocity u which the atmosphere may have is gradually destroyed by friction, not, however, at a uniform rate, since as u diminishes F_u also diminishes.

After an interchanging motion has once set in, which gives a value to u, we then have from (13_2)

$$\frac{dv}{dt} + F_v = -\cos\theta(2n+\nu)u \tag{42}$$

Hence the force which overcomes the inertia in the case of accelerated east or west velocity and the friction, represented by the first two terms of this equation, depends upon u and the earth's rotation, and when $u=0$ there is no such force and we have

$$\frac{dv}{dt} = -F_v \tag{43}$$

which indicates that any initial velocity v, not sustained by a force, is being continually reduced by friction, just as in the case of any initial velocity u as indicated by (41).

From (39) it is seen that the force which overcomes inertia and friction and gives rise to an interchanging motion between the equator and the poles depends upon A_2, which by (12) vanishes when there is uniform temperature between the equator and the poles. Without this temperature disturbance, therefore, we have $u=0$, and consequently the last member of (42) vanishes and it becomes (43). Hence without the temperature disturbance we have $u=0$ by (39), and consequently $v=0$ by (42), and by (41) and (43) if there ever were initial motions depending upon any cause these would soon be sensibly destroyed by friction. In the case of friction, therefore, and no temperature disturbance the conditions can only be satisfied by a state of rest of the atmosphere relative to the earth's surface.

154. If the whole atmosphere were at rest relative to the earth's surface and of uniform temperature we have seen that it would remain at rest. But if we now suppose the equatorial regions to have a higher temperature than the polar and that the law of variation from the pole to the equator is such that it is represented by the second inequality of (12), as is the case nearly in the mean annual temperature of the globe, this gives rise to a disturbing force expressed by the last member of (39) depending upon this inequality of temperature. In the first initial motion the air of all the strata moves toward the poles until the increase of the pressure there and the decrease in the equatorial regions gives rise to a pressure gradient and counter current in the strata below toward the equator, as has been explained. In this first initial motion of the air of the upper strata toward the poles, giving a negative value then to u in (42), the last member of this equation becomes positive and gives rise to a positive value of v or easterly motion of the atmosphere. The same can be inferred from the general principle of the effect of the earth's rotation in deflecting bodies from their course, laid down in §145. But this easterly velocity in the case of friction must be such that it increases in any latitude with altitude, but not quite so much as to satisfy (34), since if that were satisfied u in (42) would equal 0, and there would be no interchanging motion between the equator and the poles and no force to overcome the friction F_v of the easterly motion, for (34) has been obtained upon this assumption.

Whatever may be the value of v', therefore, at the earth's surface the value of v for the upper strata must be a little less than would be given by (34). If it were equal to that the deflecting force due to the earth's rotation would exactly counteract the tendency to flow toward the pole and u would vanish, and then, by (43), this velocity v would be decreased just enough to allow the air to flow toward the pole with a velocity sufficient to give a value to the last member of (42) required to overcome the friction F_v, for after the initial motions v becomes uniform, and $D_t v$ vanishes. As the friction in the upper strata of the atmosphere must be extremely small, the value of u must be very small and the value of v cannot exceed that given by (34) or fall short of it much; and the less the friction the less the difference and the less the value of u, and consequently of the interchanging motion between the equator and the poles.

In the case of no friction one of the solutions is that of § 150, in which u vanishes and v' may be equal 0, in which case we have uniform pressure from the pole to the equator at the earth's surface, or v' may have any value, since any initial value which it may have is not destroyed by friction. Where $v'=0$, the relation between the relative angular velocities of the strata at different altitudes and the absolute angular velocity of the earth's rotation is expressed by (36), and the figure which the atmosphere assumes is that of Fig. 4.

155. We have seen, § 148, that in the case of no friction with an interchanging motion between equatorial and polar regions, there is a torsional force due to the earth's rotation, which causes a very great easterly relative velocity of the air in higher latitudes and a large, but considerably smaller westerly one in the lower latitudes. The effect on atmospheric pressure and the figure which the different strata of the atmosphere of equal pressure assume in consequence of the deflecting forces arising from these large easterly and westerly velocities, is represented in Fig. 3. In the case of friction between the atmosphere and the earth's surface we of course have the same tendency, but on account of the weakness of the deflecting forces the effects are mostly counteracted by friction, so that instead of very large easterly velocities in the higher latitudes and westerly ones in the lower latitudes, as in the case of no friction, we have in this case only comparatively small ones, especially at the earth's surface, and instead of a bringing of the higher strata of equal pressure entirely down to the earth's surface near the poles and depressing them considerably at the equator, and causing a maximum pressure in the middle latitudes, we have in comparison only a very small diminution of pressure at the poles and the equator with a maximum about the parallels of 30°. Although friction tends to limit these large velocities and the effects depending upon them, yet it cannot entirely destroy or prevent them, since there must be some velocity before friction comes in play.

Since each stratum, where velocities increase from the earth's surface upward, acts through friction upon the stratum beneath, and this one upon the next below, and so on, if we put

$F_{v'}$=the force required to overcome the friction between the atmosphere and earth's surface;

m=the mass of a column of air with unit base;

we must have by (42) for each unit of surface

$$F_{v'}=\int F_v dm=-\int \cos\theta(2n+\nu)u\,dm-\int\frac{dv}{dt}dm$$

in which the integration for the column must be from the earth's surface to the top of the atmosphere. Since ν is so small in comparison with $2n$ that it may be neglected, the first term of the last member, by (40), vanishes, and we have

$$F_{v'}=-\int\frac{dv}{dt}dm \tag{44}$$

In the interchanging motions of the atmosphere, the air of the upper strata moves from the lower to the higher latitudes, where it very gradually sinks down to the lower strata, and then it returns to the lower latitudes, where it again ascends to the higher strata, thus performing a kind of circuit. Between some middle latitude and the poles the air sinks, but between that and the equator it rises. At this middle latitude there is no vertical motion down or up. In the higher latitudes where the air sinks, since the velocity is greater above than below, D_tv is negative, since the eastward velocity v gradually diminishes, but in the lower latitudes where the air is rising D_tv is positive since it is increasing. Hence in the higher latitudes $F_{v'}$ is positive, and in the lower, negative, and consequently there is a force to overcome the friction of a certain amount of velocity v', in an easterly direction in the former, and a westerly direction in the latter.

It is seen from (44) that this force arises from the moment of inertia acquired by the air in going from the lower to the higher latitudes, and *vice versa.* As the air in the higher latitudes gradually sinks this moment is lost, and is spent in overcoming the friction in these latitudes. In passing in the higher latitudes toward the poles it has at any given latitude an eastward velocity v, and on returning in the lower strata it still has an eastward velocity v', but less than the other. The momentum lost by a mass m from the time it passes the given latitude toward the pole until it returns to that latitude is $m\ (v-v')$, and hence this amount has been spent in overcoming the friction in the higher latitudes. But the amount of momentum gained in passing around toward the equator and back above to that latitude is $m(v'-v)$, which is a negative quantity, since by (34) v is greater than v' at all latitudes, and consequently this momentum has been spent in overcoming the friction of the westerly motion in the lower latitudes. The force, then, which overcomes the surface friction $F_{v'}$ (44) depends upon the difference between the easterly velocities of the upper and the lower strata, and that, by (34), upon the difference of temperature between the equator and the poles, which is $2A_2$.

156. In the case of no friction, we have seen, § 148, that the relative amounts of easterly motion in the higher latitudes and of westerly motion in the lower latitudes at the earth's surface depends entirely upon the initial motions if the initial state is not that of rest, and in case of rest they have been there determined. In the case of friction, however, they must be determined, if determined at all, upon a different principle, which is, that the sum of the moments of gyration arising from the action of the air through friction upon the earth's surface, taken over the whole surface must equal 0, else these actions would have a tendency to change the velocity of the earth's rotation, which we know can only arise from the action of external forces and not through the mutual actions of different parts of the system upon one another. Putting, therefore, σ for the earth's surface, we must have by this principle, since $r \sin \theta$ is the distance of the action from the axis, and $d\sigma = r \sin \theta\ d\theta \times 2\pi$,

$$(45) \qquad \int r \sin \theta\ F_{v'}\ d\sigma = 2\pi r^2 \int \sin^2 \theta\ F_{v'}\ d\theta = 0$$

the integral of which must be taken over the whole globe, or in the second form, from pole to pole.

On account of the general complexity of the problem, and the unknown relations between $F_{v'}$ and v', we cannot obtain any quantitative results, and we can only infer from this result that if there are easterly velocities of the atmosphere at the earth's surface in the higher latitudes, there must also be westerly velocities in the lower latitudes. As $\sin^2 \theta$ is small for the higher latitudes, the value of $F_{v'}$, and consequently of v', must be large, or else the zone of westerly velocities must be narrow and the parallel of latitude where the easterly velocity vanishes and changes sign must be not very far from the equator. Its position can only be determined from observation.

Since the friction at the earth's surface is some function which increases with the velocity, and probably very nearly in proportion, the value of v' must be such that $F_{v'}$ in (44) shall be exactly equal to the second member of that equation, and hence the greater the friction constant, that is, the amount of friction for unit of velocity, the smaller is v'. But the force $F_{v'}$, we have seen, § 155, is proportional to $2A_2$, the difference of temperature between the equator and the poles, and hence v' at the earth's surface depends likewise upon this difference of temperature as well as upon the friction constant, which varies with the nature of the surface.

The value of v', therefore, depends upon the nature of the surface. For a smooth surface, such as that of the ocean or of a smooth level plane, it is small for the same velocity v' in comparison with what it is for land surfaces with hills, forests, and mountain ranges. Taking the general surface of the two hemispheres, it is much greater for the northern than the southern hemisphere, and consequently the average or normal easterly velocities in the former are less than in the latter. Hence the great easterly velocity of the wind about the Cape of Good Hope. By (33) the greater is v' the greater is v for all altitudes, since v differs from v' by the same amount, whatever the value of the latter. Hence the value of v generally is much greater in the southern than in the northern hemisphere.

Whatever the value of v', positive or negative, the value of v, by (33), increases with increase of altitude in proportion to h, and this increase is always positive, that is, toward the east. In the lower latitudes, therefore, where v at and near the earth's surface is negative, in the upper strata it becomes positive, so that then there is an easterly motion from the pole to the equator, except exactly on the equator, where $\cos \theta = 0$, or where there is no north or south motion, and consequently u vanishes, in both of which cases, by (42), there is no force to overcome the friction of such a motion.

157. Having seen that the conditions of the problem can only be satisfied with easterly components of motion of the atmosphere at the earth's surface in the higher latitudes, and with motions having westerly components in the lower latitudes, and that these at all latitudes must increase positively with increase of altitude, so that above a certain

altitude there is an easterly component v at all latitudes, except at and very near the equator, where the deflecting forces are not sufficient to overcome the friction, it now becomes important to inquire what effect these have upon the pressure and pressure gradients of the atmosphere, both at the earth's surface and at any given altitude.

If we put

s=the resultant of the velocities u and v, Fig. 5;

i=the angle between s and u, reckoned from v toward the left;

F_s=the force required to overcome the friction in the direction of s;

we then have

$$(46) \qquad v=s\cos i \qquad F_v=F_s\cos i \qquad \frac{dv}{dt}=\frac{ds}{dt}\cos i$$

$$u=-s\sin i \qquad F_u=-F_s\sin i \qquad \frac{du}{dt}=-\frac{ds}{dt}\sin i$$

The last two of these relations are upon the hypothesis that i remains constant with regard to t, which is not strictly correct, but the inertia terms are so small that this hypothesis cannot lead to much error.

From these equations we get

$$(47) \qquad \frac{du}{dt}+F_u=-\left(\frac{ds}{dt}+F_s\right)\sin i$$

$$\frac{dv}{dt}+F_v=\left(\frac{ds}{dt}+F_s\right)\cos i$$

and from these and (42) we get

$$(48) \qquad \tan i=\frac{\frac{du}{dt}+F_u}{\cos\theta(2n+\nu)u}=\frac{\frac{ds}{dt}+F_s}{\cos\theta(2n+\nu)s}$$

dividing the first form of the expression by $u:s=-\sin i$ in order to obtain the last form.

Where the frictional resistances can be regarded as a function of the velocity and proportional to the first power of the velocity, as it may very nearly at the earth's surface, we can put $F_s=fs$, f being an unknown friction constant. The preceding equation thus becomes, by neglecting D_ts in comparison with F_s

$$(49) \qquad \tan i=\frac{f}{\cos\theta(2n+\nu)}$$

At the earth's surface where the friction is large, the term D_ts, which expresses the inertia of the air in varying velocities, and vanishes when the velocity is constant, must be very small, and the inclination i must depend almost entirely upon the friction.

From (9_1) we get

$$(50) \qquad \frac{d\log P}{a\,dU}=\cos\theta(2n+\nu)v-\left(\frac{du}{dt}+F_u\right)$$

From (47) we get

$$-\left(\frac{du}{dt}+F_u\right)=\left(\frac{ds}{dt}+F_s\right)\sin i=\left(\frac{dv}{dt}+F_v\right)\frac{\sin i}{\cos i}=-\cos\theta(2n+\nu)u\tan i$$

using (42) in obtaining the last form of the expression.

By means of this and the expressions of u and v in (46) the preceding equation becomes

$$(51)\quad \frac{d\log P}{a\,dU}=\cos\,\theta(2n+\nu)s\cos i+\cos\,\theta(2n+\nu)s\sin i\tan i$$

$$=\frac{s\cos\,\theta(2n+\nu)}{\cos i}=\frac{v\cos\,\theta(2n+\nu)}{\cos^2 i}$$

This expression is in terms of the resultant velocity s and the inclination i from the parallel of latitude, which, we have seen, depends upon the friction and inertia of the air. Where these can be neglected we have, by (46), $s=v$, and the preceding expression is reduced to (9_1) if we neglect the friction and inertia terms in the second member. Where $s\cos i=v$ is positive the pressure increases from the poles towards the equator, and the reverse where it is negative.

In Fig. 5 let $AB=s$ represent the resultant velocity of the wind inclined to the component $AD=v$ by the angle $BAC=i$. In the case of friction the increase of pressure from the pole towards the equator depends upon the deflecting force arising, by (50), from the easterly component of motion v, represented by AD in Fig. 5, and upon the inertia and friction of the air represented by the last two terms in the expression of (50). Since $AC=s:\cos i$ in (51) the line AC represents the easterly or westerly velocity required to give the same value to the first member of (51) in case of no friction and inertia, in which case, we have seen, i vanishes and s, AC and AD in Fig. 5 become the same. Where there is a component of motion DB or $-u$ toward the north pole, we must have a less deflective force toward the equator, than in the case of no friction and inertia, so as not wholly to counteract the tendency to flow toward the pole which arises from the decline in the strata of equal pressure toward the pole, as represented in Fig. 4, and to leave a part of this force to overcome the friction to the component of motion $-u$ toward the pole. Consequently we must have an easterly velocity $v=AD$, less than AC, and the deflecting force belonging to the part CD is the measure of the force required to overcome the friction and inertia of the component of motion DB.

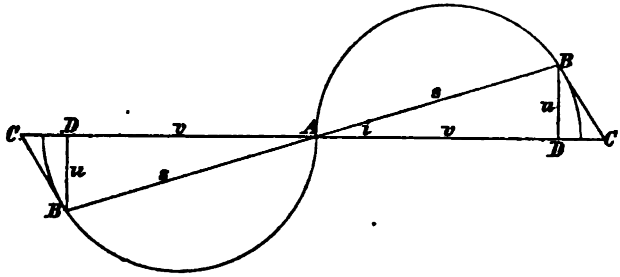


FIG. 5.

Where the one component is west or negative, as represented by AD to the left, the other, at the earth's surface at least where the friction increases with the velocity, is necessarily south or positive, and i then becomes greater than 180°. In this case the pressure gradient is negative, that is, the pressure decreases toward the equator.

158. If we put

$$(52) \qquad \frac{dP}{dU}=\frac{\delta P}{\delta U}=\frac{G}{111111111}$$

then G is the finite variation in millimeters of the pressure P, in the direction of U, in the distance of one degree of the meridian, which is 111111111 millimeters. Where the pressure is measured in millimeters of mercury G is called *the barometric gradient.* In this it is supposed that the pressure varies with U at the same rate throughout the unit of one degree. With this (51) gives, since d log P is equal $dP:P$, by putting for α its value in (4)

$$G=111111111\frac{dP}{dU}=\frac{111111111}{gl(1+.004\tau)}\cdot\frac{s\cos\theta\,(2n+\nu)}{\cos i}P$$

Putting $P_0=0.760^{\mathrm{m}}$ this gives with the value of gl in Table XIV,

$$(53) \qquad G=\frac{1077.4\cos\theta\,(2n+\nu)}{\cos i\,(1+.004\tau)}\cdot\frac{sP}{P_0}$$

Near sea-level the last factor can generally be put equal unity without much error.

Since ν is very small in comparison with $2n$, especially near the earth's surface, we can generally neglect it without any sensible error, and then with the value of ν in Table XIV, we get

$$(54) \qquad G=\frac{.1571\,s\cos\theta}{\cos i\,(1+.004\tau)}\cdot\frac{P}{P_0}=\frac{.1571\,v\cos\theta}{\cos^2 i(1+.004\tau)}\cdot\frac{P}{P_0}$$

This is the general expression of the barometric gradient as measured on the meridian from north to south. In the southern hemisphere $\cos\theta$ becomes negative and consequently the sign of the gradient is reversed, but is the same if reckoned from the south pole toward the equator. Where also the component v is negative, or toward the west, the gradient, by (54), is negative in the northern hemisphere and the contrary in the southern. Wherever, therefore, in the northern hemisphere there is a westerly component of motion at any altitude there is a negative gradient and the pressure decreases toward the equator.

The gradient at any latitude also decreases with decrease of P, that is, with increase of altitude, for the same velocities of motion s or v, the inclination i remaining the same. This arises from the diminution of density which is as P where the temperature is the same. For the same east or west components of velocity, the decline in the strata of equal density, Fig. 4, is the same, and the increase or decrease of gradient depends upon the weight of air between the stratum of equal pressure be and the horizontal stratum bc, and therefore is the less the greater the altitude, since the greater the altitude the less the density.

Where the gradients instead of the velocities are known, (54) can be reversed and the velocities computed from the gradients, if the inclination i is also known. As this depends mostly upon the uncertain element of friction, it cannot be determined generally except by observation, and this for the most part only at and near the earth's surface. The barometric gradients also are known at the earth's surface only. A formula, therefore, for computing air velocities from the observed gradients at the earth's surface may be obtained from (54) by putting them $P:P_0=1$, as it is very nearly, and we thus get

$$(55)\qquad s=\frac{\cos i(1+.004\,\tau)\,G}{.1571\cos\theta}$$

Where i is not known, we can put without much error $\cos i=1$, since i is not so large, generally, that the cosine differs much from unity. By doing this we get AC, Fig. 5, instead of s.

Near the equator, however, the preceding formulæ practically fail. On account of the small value of $\cos\theta$ in (48) and (49) the value of i there becomes large, and consequently, where it is not accurately known from observation, the uncertainty in the formulæ is very much increased, since a small error in the value of i then affects the value of the cosine much more than when i is small. And this uncertainty is especially great in (55), in which it is magnified by the smallness of $\cos\theta$ in the denominator. We cannot, therefore, obtain velocities from the gradients by (54) with much accuracy near the equator.

159. If in (54) we substitute for v its value in (34) at the earth's surface where $h'=0$, we get, using the numerical values of the constant in Table XIV in the reductions,

$$(56)\qquad G=G'-\frac{.00001328\,hA_2\sin 2\theta}{1+.004\,\tau}\cdot\frac{P}{P_0}$$

In which, assuming $P'=P_0$,

$$(57)\qquad G'=\frac{.1571\,S'\cos\theta}{\cos i(1+.004\,\tau)}=\frac{.1571\,v'\cos\theta}{\cos^2 i(1+.004\,\tau)}$$

Hence, where there is an east component of velocity at the earth's surface the gradient is positive, but where there is a west or negative component it is negative, and where there is neither an east nor west component there is no gradient. In the latter case we have the condition of the atmosphere represented in Fig. 4, in which the gradient above the earth's surface depends upon the unequal raising of the strata of equal pressure by the greater temperature expansion in the equatorial than in the polar regions.

In using the expression of (34) above it is assumed that in the upper strata the value of i is so small that v is the same as it would be in the case of no friction, or, that $AD=AC$ in Fig. 5. This is evidently so nearly so at only a small altitude, that it can be so assumed without any sensible error.

In (54) the value of P for any given altitude h, can be obtained with sufficient accuracy from the table of § 27.

We have seen, § 155, that in consequence of the interchanging motion of the atmosphere between the equatorial and polar regions of the earth there is necessarily an easterly component of motion at the earth's surface in the higher latitudes and the contrary in the lower latitudes. Hence in the northern hemisphere the gradient by (57) is positive in the higher latitudes and negative in the lower ones, and greater or less in proportion to v' where the volume of i is the same for all velocities, as it is by (49) where friction is as the velocity. The gradient, therefore, must be greater in the southern than in the northern hemisphere, since v', § 156, is greater in the former than in the latter.

It is seen, (57), that the gradient at the earth's surface depends entirely upon the surface velocity s' and corresponding inclination i, and where v' vanishes there is no gradient. Whatever the value of v', there is, by (34), the same difference between this and the values of v above, so that if the value of v' is increased all the easterly components of motion v above are increased just as much, and in the lower latitudes where v' becomes negative, the values of v above, considered algebraically, are just as much less. With the positive values of v' in the higher latitudes, the values of v at all altitudes are increased just as much above what they would be in case of $v'=0$ and no gradient, and the deflective force, §145, arising from this increase of the easterly component of velocity tends to drive the air from the poles toward the equator and to produce a gradient, and a lowering of pressure at the earth's surface in the polar regions. Likewise the deflective force arising from the decrease of velocity in the lower latitudes below what it would have in case of no east or west component of velocity at the earth's surface and no gradient there, being negative, has the contrary effect, and causes a gradient of pressure increasing from the equator toward the poles. The easterly component of velocity which the atmosphere has at all altitudes and latitudes in case of no east or west component at the surface, we have seen, has no effect upon the gradients at the surface.

In the real case of nature, therefore, in which there are necessarily easterly components of motion in the higher, and westerly ones in the lower latitudes, we must have a diminution of atmosphere and of pressure in the polar regions, an accumulation and increase of pressure in the middle latitudes, and again a smaller diminution of atmosphere and of atmospheric pressure in the equatorial regions, and the strata of equal pressure at and near the earth's surface will have the form somewhat as represented in Fig. 6, with a maximum pressure at *e*, and not be as in Fig. 4 in the hypothetical case of §150. There is a slight approximation toward the results of the case without friction, §149, as represented in Fig. 3, but instead of the very great east and west components of velocity in that case, we have in the case of nature comparatively very small ones, and consequently a very small variation of pressure in comparison between the equator and the pole.

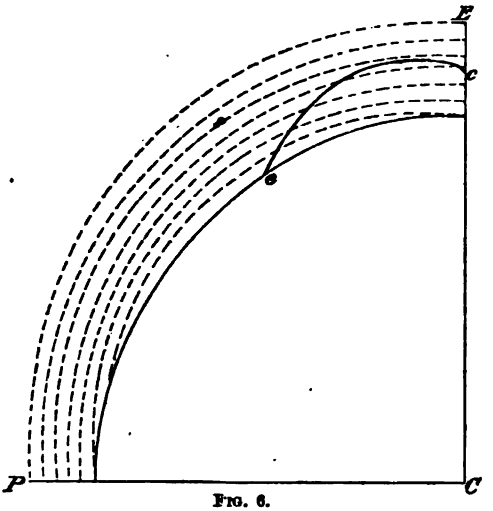


FIG. 6.

160. From (13_1) we get in case of the mean temperature of the earth, using only the principle component in the last term, of which the characteristic is $s=2$,

$$v=\frac{d\log P'}{\cos\theta(2n+\nu)\alpha dU}-\frac{.008ghA_2\sin 2\theta}{\cos\theta(2n+\nu)r(1+.004\tau)}+\frac{\frac{du}{dt}+F_u}{\cos\theta(2n+\nu)}$$

The last term depends upon the inertia and friction of the meridional motions of the atmosphere, mostly upon the latter, the effect of which even at the earth's surface is small, and in the upper regions of the atmosphere it is so small that it may be neglected without sensible error.

By means of (52) and (4) and the values of the constants in App. Table XIV, we get, by neglecting ν in comparison with $2n$ and putting $P'=0.76^m$, as we can without sensible error,

$$\frac{d\log P'}{\cos\theta(2n+\nu)ad U}=\frac{dP'}{dU}\cdot\frac{gl(1+.004\tau)}{\cos\theta(2n+\nu)P'}$$
$$=\frac{G'}{11111111}\cdot\frac{gl(1+.004\tau)}{\cos\theta(2n+\nu)P'}=\frac{6.37G'}{\cos\theta}$$

Also, by putting $\sin 2\theta=2\sin\theta\cos\theta$, we get

$$-\frac{.008ghA_2\sin 2\theta}{\cos\theta(2n+\nu)r(1+.004\tau)}=Kh$$

in which, neglecting ν and using the values of the constants as above,

$$(58)\qquad K=-\frac{.0001690A_2\sin\theta}{1+.004\tau}$$

Hence the preceding expression of v becomes, neglecting the last term,

$$(59)\qquad v=\frac{6.37(1+.004\tau)G'}{\cos\theta}+Kh$$

The value of τ in (58) for the earth's surface is given by (12) for any polar distance θ, but it can be obtained more conveniently from the Table .... of § 116. A more suitable value, however, would be that of the mean temperature of the air column of the altitude h. The effect, however, of a small variation in the value of τ is so small that we can use the surface temperature without any material error.

The last term in (59) is the same as the second member of (34), and at the earth's surface where $h=0$, (59) is the expression of v', the value of v at the earth's surface where we can neglect the friction and inertia. This, however, at and near the earth's surface cannot be neglected except in rough approximate results, but the effect of the friction, we have seen, is in causing an inclination of direction, i, from the tangent to the isobars, and where this is known from observation, the value of v' or of s' belonging to any gradient G' at the earth's surface, can be obtained from (57). But for any altitude a little above the earth's surface, and especially for great altitudes, (59) must give the value of v without any sensible error except at and near the equator where cos θ vanishes or becomes so small that it may make the effect of any small uncertainty in the value of G', and also the effect of the neglected friction term in the expression of v, which likewise has cos θ in the denominator, very large. We cannot therefore use (59) very near the equator without incurring the risk of large errors.

The effect of the neglected friction and inertia term upon the value of v, it is seen, is positive where the direction of meridional component of motion or u is positive, and hence in the upper strata of the northern hemisphere where u is negative, the value of v given by (59) is a very little too great, while in the lower strata, and especially in the trade-wind zone at the earth's surface, the value of v is too small taken algebraically, and consequently too large regarded negatively where it has a negative value. In the middle latitudes, however, near the earth's surface, where u is negative, the value of v is less than that given by (59). In the southern hemisphere the values of both u and cos θ change signs, and hence we have the same effects there on the same parallels of latitude upon the value of v.

By (54) the gradient vanishes, and we consequently have the maximum pressure, where $v=0$. Hence the maximum pressure at e, Fig. 6, at the earth's surface is where v' vanishes and changes sign. Since in the higher parallels of latitude the value of v' is necessarily positive and in the lower negative, by (57) there is a corresponding change in the sign of G'. Since the last term in the expression of (59) is always positive, the value of A_2 in the expression of (58) being negative, where G' is positive v cannot vanish and become negative at any altitude, and hence there can be no maximum pressure in any horizontal stratum above in any latitude where G' is positive. But between e and the equator, Fig. 6, where G' is negative, v vanishes at a certain altitude depending upon the value of G' and of K in (59). The value of G' vanishes at or very near the equator, and also at *e*, and is consequently greatest at some intermediate latitude between. The greater the negative value of G' the greater the altitude at which v vanishes and at which the pressure is a maximum, and nearer the equator where G' is less, the altitude at which v vanishes and at which the pressure is a maximum is less. If we let the curved line *ec*, Fig. 6, represent the altitude at different latitudes at which v vanishes, this will be the highest very nearly where the negative values of G' are the greatest, since K in (59) in these latitudes is nearly constant, and it will not be so high nearer the equator. At every point of *ec*, v vanishes and changes sign and the pressure is a maximum in going horizontally from the pole toward the equator. Hence at certain altitudes within the tropics there may be two maxima, with a minimum between, this latter occurring where G' has its greatest negative value. At all altitudes below the curved line *ec* the value of v is negative.

161. The normals of latitude for the pressure, and from these the normal gradients of latitude, have been approximately obtained from observation, just as the normals of temperature and the temperature gradients have. These normal pressures reduced to standard gravity, and the corresponding gradients deduced from them by means of the differences, are given in millimeters for the mean of the year and for January and July in the following table for each fifth degree of latitude:[2]

Latitude.	Northern hemisphere.						Southern hemisphere.					
	Annual mean.		January.		July.		Annual mean.		January.		July.	
	P'.	G'.	P'.	G'.	P'.	G'.	P'.	G'.	P'.	G'.	P'.	G'.
° 80	760.5	....	760.4	....	700.6	....	....	....	....	....	....	....
75	760.0	—0.19	760.2	—0.15	758.8	—0.23	....	....	....	....	....	....
70	758.6	—0.14	759.0	—0.09	758.2	0.19	738.0	....	....	....	....	....
65	758.2	+0.01	758.8	+0.07	757.6	—0.05	730.7	—0.65	....	....	....	....
60	758.7	0.15	759.7	0.21	757.7	+0.09	743.4	0.83	....	....	....	....
55	759.7	0.20	761.0	0.25	758.4	0.15	748.2	0.97	748.2	—0.87	748.2	—1.07
50	760.7	0.18	762.1	0.21	759.3	0.15	753.2	0.91	752.7	0.81	753.7	1.01
45	761.5	0.15	763.0	0.17	760.0	0.13	757.3	0.73	756.3	0.64	758.3	0.82
40	762.0	+0.07	763.6	+0.08	700.4	+0.06	760.5	0.51	759.1	0.44	761.9	0.58
35	762.4	—0.03	764.1	—0.03	760.7	—0.03	762.4	0.30	700.6	0.25	764.2	0.35
30	761.7	0.18	763.4	0.19	760.0	0.17	763.5	—0.08	761.3	—0.05	765.7	—0.11
25	760.4	0.25	762.0	0.28	758.8	0.22	763.2	+0.18	760.8	+0.18	765.6	+0.18
20	750.2	0.21	760.6	0.27	757.8	0.15	761.7	0.29	759.5	0.26	763.0	0.32
15	758.3	0.13	759.3	0.22	757.3	—0.04	760.2	0.26	758.2	0.20	762.2	0.32
10	757.9	—0.03	758.4	0.14	757.4	+0.08	759.1	0.20	757.4	+0.12	760.8	0.28
5	758.0	+0.01	758.0	0.11	757.9	0.13	758.3	0.11	757.1	0.00	759.5	0.22
0	758.0	+0.04	757.4	—0.08	758.6	0.16	758.0	+0.04	757.4	—0.08	758.6	+0.16

It is with the annual means only that we have to do here. Using the values of G' for these means, we can compute in (59) the approximate normal values of the first term or v' for each latitude or polar distance, and with the values of this first term, and the value of K for any latitude, the value of v can be conveniently obtained for any altitude and any latitude. The following table contains the values of this first term and also of K in (59), for each fifth degree of polar distance, except near the equator, where (59) in a great measure fails for reasons already given, and near the poles, where the data from observation are not reliable. In the computation the value of $1+.004\tau$ was assumed to be equal to $1-.084 \cos \theta$, neglecting the constant term in (12), thus making the mean temperature on the parallel of 45° equal to 0°. As we are considering here the effect of the second and principal inequality only in (12) the others are neglected.

The formula of course gives the results in meters per second since these have been the units adopted; but in the following table they are given in kilometers per hour, and the values of the constant K are such that h in the table must be taken in kilometers:

Latitude.	K.	Northern hemisphere.		Southern hemisphere.	
		v'	$v(h=5\text{km})$	v'	$v(h=5\text{km})$
°					
75	3.6	−4.2	13.8	....	....
70	4.7	−3.1	20.4	....	....
65	5.8	+0.2	29.2	+15.4	44.4
60	6.8	3.7	37.7	20.7	54.7
55	7.7	5.4	43.9	26.0	64.5
50	8.5	5.2	47.7	26.4	68.9
45	9.2	4.7	50.7	23.2	69.2
40	9.7	+2.5	51.0	18.1	66.6
35	10.1	−1.2	49.3	12.2	62.7
30	10.6	8.4	44.6	+ 3.8	56.8
25	11.0	14.2	40.8	−10.2	44.8
20	11.3	15.0	41.5	20.5	36.0
15	11.5	12.3	45.2	−24.1	33.4

If the effect of the other small terms in (12) and in (33) had been taken into account the values of the first term of (59), and especially of K above, would have been somewhat different, but the differences are of no importance in approximate results of this kind where the uncertainties in the data are considerable.

162. The values of G' within the north polar circle, and consequently the values of v' by (57), it is seen in the table, are negative. There does not seem to be any explanation of this in theory, for by §159 the force which causes an eastward component of motion extends to the poles.

There is considerable uncertainty in the data from which the values of P' and G', in the preceding table, were obtained for these high latitudes, and small errors may have reversed the sign of the gradients in the table where these negative values have been obtained. Between the polar circle and the parallel of 36° in the northern hemisphere, and south of 29° in latitude in the southern hemisphere, as far as barometric observations have been made, the gradients, taken in in each hemisphere from the pole toward the equator, are positive, and hence by (57) the values of v' are positive, and in the southern hemisphere where the gradients are steep these values are very large. By a glance at Plate III of Coffin's "Winds of the Globe,"[3] it is seen that this is exactly in accordance with observation, except that the dividing line between the two systems of winds in the southern hemisphere is a little south of the parallel of 29°. But the observations in this zone are not sufficient for determining the position of this line very accurately, and the position obtained from the barometric gradients is, without doubt, the more correct one.

According to Mr. Laughton, also, in his excellent description of the winds,[4] they have a strong, easterly component of motion in the middle latitudes. "In both hemispheres to the north and south of the parallels of 35° or 40°, a strong westerly wind blows with great constancy all around the world. In the southern hemisphere, more particularly, it blows with a persistence little less than that of the trade-winds, but with a strength which, although fitful, is very much greater. From a fresh, strong breeze it rises frequently into a violent gale, and as such blows for days together, the mean direction being nearly west, from which it seldom varies more than a couple of points on either side. South of the Atlantic, south of the Indian Ocean, south of Australia, in the higher latitudes of the Southern Pacific, and to the south of Cape Horn, we find it still the same, a westerly gale, whose strength and constancy combined have enabled Australian clippers to make passages which seem to border on the fabulous. In the northern hemisphere it has not the clear range which it has in the southern; but there, too, it prevails in the most decided manner."

This description of the winds in the middle latitudes of both hemispheres accords exactly with the results in the preceding table obtained theoretically from the normal gradients of the observed pressure at the earth's surface in all latitudes. The computed values of the first term of v are small in the middle latitudes of the northern hemisphere, indicating that the east component of normal wind velocity at and near the earth's surface is small there; but in the same latitudes of the southern hemisphere these computed components of velocity are comparatively very large, and hence the persistent strong westerly winds all around the globe on and near the parallel of Cape Horn. The computed values of the first term of v, (59), on the parallel of 50° south is 26.4^{km} (16.4 miles) per hour for the average velocity, as deduced from the gradients of the preceding table. This cannot vary much from the true surface value of v, since the effect of friction on the water surface of the southern hemisphere and hence the value of i is consequently small, in which case cos i is sensibly unity in (57) and v in (59) equal to v' where $h=0$. The reason of the greater strength of the normal westerly winds here than on the same latitudes of the northern hemisphere has been given in § 156.

163. The computed westerly components of velocity in the trade-wind zones are greater in the southern than in the northern hemisphere. This is in accordance with observations of the trade-winds of the Atlantic Ocean, which show that the strength of the southeast trade-winds is considerably greater than that of the northeast ones. The theoretical results, however, in these latitudes are not very reliable, since it was assumed that $i=0$ in (57), but in these latitudes the value of i is such as to make this assumption erroneous, especially very near the equator, as may be seen from (48) and (49), the value of $\cos\theta$ here being such as to give a larger value to i. For this reason the computations in the preceding table are not extended nearer to the equator than the parallel of 15°.

The westerly components, however, between the tropics and the equator are verified by observations all around the globe in both hemispheres, but most especially on the oceans. In the North Atlantic Ocean the northeast trade-wind blows with almost uniform steadiness between the parallels of 25° and 10°, and hence has a westerly component, though somewhat less, probably, than what is given in the preceding table, as it should be, since the value of i here is such as to make $\cos i$ in (57) much less than unity, and hence v' should be less here than the value computed with $\cos i=1$. Across the entire Pacific Ocean, in a zone of about the same width but with limits a little nearer the equator, the northeast trade winds blow with great regularity, and hence there is here the same westerly component of the winds. South of the equator, but in general a little nearer to it, in both oceans, is the similar system of southeast trade-winds, in which, likewise, the winds have westerly components of velocity, but less, as they should be, than those given in the table.

The same system prevails also, but with less regularity, over the continents in both hemispheres. The few observations in the interior of North Africa indicate that the prevailing winds are from the east and northeast. "The sand which these winds raise in the desert is carried by them far out to sea. During the summer it fills the air as far as the Canaries and to the height of the Peak of Teneriffe with an impalpable dust, which has the effect of a thick haze; and at all seasons of the year ships, even at a considerable distance from the shore, find it covering their sails and decks."[4]

On the same latitudes in the interior of Africa south of the equator Livingstone and other observers have found the prevailing winds to be from the east and southeast, and in Brazil at certain seasons, even up to the Amazon, it is well known that easterly winds prevail, extending far into the interior, if not even to the base of the Andes.

164. With regard to atmospheric currents at considerable altitudes, it is seen that, according to the preceding table, they must have a large easterly component of velocity, and the greater the altitude the greater

must be this component. In the northern hemisphere, on the parallel of 45°, at the height of 5^{km} (3.1 miles) it is nearly 51^{km} (31.6 miles) per hour, while in the southern hemisphere on the same parallel, on account of the greater easterly component of velocity at the surface of the earth, it is 69^{km} at this altitude. These are very nearly the maxima of latitude; nearer the poles they diminish on account of the smaller value of the constant K in (57), and nearer the equator on account of the westerly component at the earth's surface.

On the parallel of 25° in the northern hemisphere we have, at the altitude of 5^{km}, $v=40.8^{km}$, but at a lower altitude it must vanish and change sign. If from the condition formed from (58) by putting $v=0$, using the values of v' and K in the preceding table, we determine the values of h which satisfy this condition, we get, in kilometers, h equal 0.1, 0.8, 1.3, 1.33, 1.07, for the altitudes at which $v=0$, on the latitudes, respectively, of 35°, 30°, 25°, 20°, and 15°. Hence the altitudes at which there is no east or west component of velocity increase at first from the latitude where $v'=0$ down towards the equator, and then they seem to decrease, as represented in Fig. 6. The true altitudes, however, are no doubt considerably greater, since these computations are based upon the theoretical values of v' in the table, which, for reasons already given, are no doubt too large. At any rate, within the tropics where there is a considerable westerly component of velocity at the earth's surface, at an altitude of about a mile at most, this must be reversed and become an easterly one for all greater altitudes and increase in proportion to the increase of altitude.

On account of the small amount of friction in the upper strata of the atmosphere, the value of i by (48) must be very small, and the easterly components of velocity may be regarded as being the true velocity of resultant motion.

165. The preceding deductions from theory with regard to the motions of the upper currents are confirmed by observations in all parts of the world where observations have been made. Strong westerly winds have been observed on Mauna Loa in the Sandwich Islands, on the passes of the Rocky Mountains and the Andes, on the top of Pike's Peak and Mount Washington, on the Peak of Teneriffe, and at every elevated position in either hemisphere all around the globe, except in the calm-belt near the equator, even where there is an easterly wind in the same latitude at less elevated positions. On the 1st of May, 1812, the island of Barbadoes, within the trade-wind zone, where there is a westerly component of the wind, was suddenly obscured by a dense cloud and its surface covered with ashes from an eruptive volcano of St. Vincent, more than a hundred miles toward the west. Also, on the 20th of January, 1835, the volcano of Coseguina, on the Lake of Nicaragua, lying in the belt of the northeast trade-winds, sent forth great quantities of lava and ashes, and the latter were borne in a direction just

contrary to the surface wind and lodged on the island of Jamaica, 800 miles to the northeast. These instances indicate that within the tropics where there is a westerly component of the winds at the earth's surface, the currents above at no great elevation are such as to carry the ashes of volcanoes eastward.

No one can fail to observe that the general direction of the higher clouds in fair weather is easterly, and especially in the case of the very high cirrus clouds. The easterly velocities of these latter have been estimated by Rev. Clement Ley to be at times as much as 120 miles, and it rarely happens that they have no easterly tendency. The average, therefore, of these somewhat uncertain estimates may be put at nearly 100 kilometers. This is about the velocity given by the data in the preceding table in the middle latitudes of the northern hemisphere at an altitude of 10^{km}, which is about that of the cirrus clouds.

The easterly motion of the atmosphere above within the tropics is also confirmed by observations made on the directions of the clouds at Colonia Tovar, Venezuela, latitude 10″ 26′. According to these observations, while the motion of the lower clouds was in general from some point toward the east that of the higher clouds was from some point toward the west.

In the comparison of observations with the theoretical results of the preceding pages it must be remembered that the observed velocities and directions are affected by numerous abnormal disturbances, while those of theory are the normal ones simply, belonging to the normal inequality of temperature between the equatorial and polar regions. Only the general tendency or average of observations, therefore, can be used in the comparisons.

166. The maximum pressure at the earth's surface, at e, Fig. 6, which in the northern hemisphere by observation is near the parallel of 35°, but in the southern hemisphere a little nearer the equator, as is seen from the values of P' in table, § 161, causes the air to flow out from beneath on both sides; on the equatorial side this combines with the motion in the lower part of the atmosphere from the polar to the equatorial regions, and the resultant, deflected westward by the influence of the earth's rotation, gives rise to the northeast and southeast trade-winds; but on the polar side the tendency to flow out from beneath toward the poles is so great that it more than counteracts the tendency to flow from the poles toward the equator, and the resultant, likewise deflected by the influence of the earth's rotation, gives rise to the gentle normal southwest winds of the northern hemisphere and northwest winds of the southern hemisphere, which prevail in the middle latitudes. Without a component of motion toward the equator on the one side, we should have $u=0$, or negative, and hence by (42) there would be no force to overcome the friction of a westerly component of motion, for this would require a negative force, and without a negative meridional com-

ponent of motion on the other side, we should by (42) have no force to overcome the resistance to an easterly component of motion. By means of this underflow, therefore, from beneath this maximum pressure we get forces, arising from the influence of the earth's rotation, which are necessary to overcome the frictional resistance to the westerly component of velocity on the one side, and the easterly component on the other side, of this parallel of greatest pressure.

At or near the equator the counter currents from the poles toward the equator of the lower strata of the two hemispheres meet and we have $u=0$. Hence by (42) there is no force there to overcome the frictional resistance to an east or west component of motion, and consequently we also have $v=0$. There is therefore a perfect calm at or near the equator, where there is no motion north or south in the direction of the meridian.

On the parallels of greatest pressure, also, since the air flows out from beneath this pressure both toward the equator and the pole, there is no motion in the meridian, and we consequently have here also $u=0$, and then by (42) also $v=0$, and consequently a perfect calm. At the poles, also, we necessarily have $u=0$, and hence a calm; for u cannot have much value for some distance from the poles.

There are, therefore, so far as concerns the normal motions of the atmosphere, three calm belts extending entirely around the globe: One a little north of the equator, and the others, one in each hemisphere at the parallels of maximum pressure at the earth's surface, on the parallel of 35° in the northern hemisphere and on that of 30° in the other. The mean position of the equatorial calm-belt is about 5° north latitude in both the Atlantic and Pacific Oceans, and mean breadth about 6°, but in summer it is more than twice as wide as in winter. The unsymmetrical positions of these calm-belts with reference to the two hemispheres is due, in a small measure, to the unsymmetrical distribution of temperature in the two, which, we have seen, § 116, causes the normal thermal equator to be a little north of the true equator; but it is due mostly to the greater velocities of the general motions of the atmosphere in the southern hemisphere, as seen in the results of the table in § 161, the deflecting forces of which tend a little to drive the air from the southern into the northern hemisphere, and also to move all the calm-belts a little from the south towards the north.

From what precedes, therefore, the general normal motions of the atmosphere, both on the earth's surface and in vertical meridional section, are somewhat as represented in Fig. 7, in which the arrows show the directions of the winds, and the stars the positions of the equatorial and tropical calm-belts and the polar calms.

The mean position of the equatorial calm-belt is about 5° north latitude in both the Atlantic and Pacific Oceans, and mean breadth about 6°, but in summer it it is more than twice as wide as in winter,

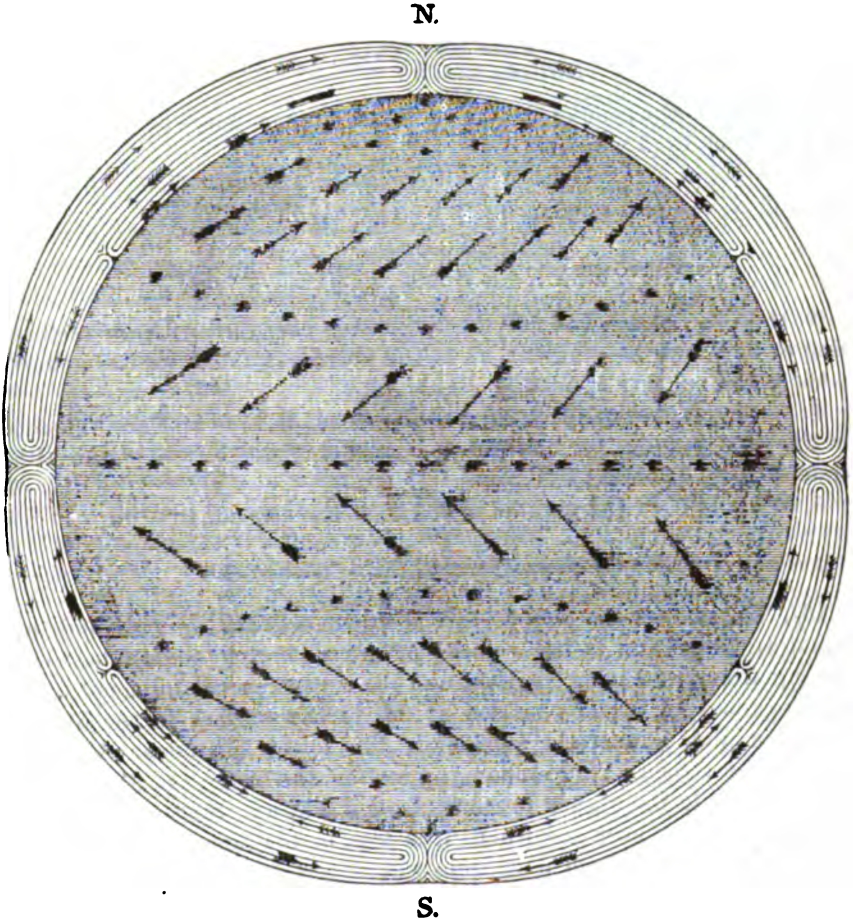


FIG. 7.

It must not be understood, however, that these belts actually exist all around the globe, but only that they would so exist if there were no abnormal variations of temperature from the normal temperature of latitude, and the surface of the earth were homogeneous all around. On account of the various disturbances arising from the abnormal variations of temperature, and from the irregularities of land and water surface and mountain ranges, the regularity of these belts is very much broken up, and is observable mostly on the oceans only.

Examples.

1. The barometric gradient at the surface of the earth, G', by the preceding table, for the annual mean in the northern hemisphere on the parallel of 45°, is 0.15^{mm}. What is the value of v', supposing the inclination i to be 30° and $\tau = 0$?

By (57) we have, in miles per hour, Table XIV,

$$v'=\frac{\overline{0.15\times\cos^2 30^\circ}}{.157\cos 45^\circ}\times 2.237=2.27 \text{ miles.}$$

2. What is the gradient under the same circumstances at the altitude of 8 kilometers (5 miles)?

From table, § 27, we have approximately $P=284$. Hence by (56), putting $A_2=-21^\circ$, we get, $\sin 2\theta$ being unity in this case,

$$G=0.15+\frac{.00001328\times 8000\times 21\times 284}{760}=0.98^{\text{mm}}$$

3. If on the parallel of 55° the air moves eastward with an inclination of 35°, and the gradient, as in the preceding table, for 55° is 0.20, what is the velocity s_1, putting $\tau=0$? (55)

4. By the last of the preceding tables, what is the eastward velocity of the wind in miles per hour on the parallel of 60° N. and at the altitude of 10^{km}, Table XIV?

5. What is it on the parallel of 25° S. at the altitude of 3^{km}?

For the annual inequalities of temperature.

167. The greater the temperature gradient between the pole and the equator, the greater is the velocity of interchanging motion between the equatorial and polar regions, and the greater is the torsional force which causes an easterly motion at the earth's surface in the higher latitudes and a westerly motion in the lower latitudes. For the greater the velocity of interchanging motion, the faster a particle of air settles down in the higher latitudes from a stratum where the velocity v is greater to a lower one where it is less, or the faster it rises up in the lower latitudes from a stratum where it is less to one where it is greater, and consequently the greater is the value of $D_t v$. The greater, therefore, is the difference of temperature between the equator and the poles, the greater is the value of the second member of (44), and consequently the greater is the force which overcomes the resistances to east or west motions at the earth's surface. The small east and west velocities of motion at the earth's surface in all latitudes are therefore increased in winter a little above the mean and decreased in summer a little below it, for by the table of § 116 the difference between the equatorial and polar temperature in the northern hemisphere is about 40° greater in winter than in summer, the mean difference being 42°. In the southern hemisphere, however, these annual variations of temperature are comparatively very small, but the resistances to the east and west motions by the water surface there being much less, the effects of the annual changes are about of the same order there as in the northern hemisphere.

The east and west motions at the earth's surface being the greatest during the midwinter of each hemisphere and the least during midsummer, by (57) there must be a corresponding variation in the gradients between winter and summer, and observation shows this to be the case, for by comparing the January with the July gradients in a preceding table, it is seen that in all latitudes except the north polar latitudes, where the gradients are very uncertain, and at the calm-belt of the equator, the winter gradients of each hemisphere are considerably the greater. The positive gradients, then, in the higher, and the negative gradients of the lower, latitudes of the northern hemisphere, and the contrary in the southern, being both greater in winter than in summer, there is an increased pressure in winter over that of summer and an increased bulging up of the strata of equal pressure in the middle latitudes, as is shown by observation to be the case, by a comparison of winter with summer pressures in the table.

The increased activity of motions in each hemisphere in winter has no tendency to drive the air out of it, and the contractions of volume of air from decrease of temperature depresses the upper strata of equal pressure in the winter hemisphere, and the simultaneous upward expansion of the air in the opposite hemisphere raises these strata there, so that there is then a tendency of the air above to run from the hemisphere of higher temperature to the one of lower temperature. Hence each hemisphere has a little more air in it, by weight, in winter than in summer, though less by volume. There is, therefore, an annual inequality of the normal pressures of latitude corresponding to that of the temperature, with its maximum in midwinter of each hemisphere.

168. In the interchanging motions between the equatorial and polar regions we have seen that the tendency of the upper part of the atmosphere to flow toward the poles is checked and almost entirely counteracted by the increased easterly motions of the air above, so that there is only a very slow interchange in comparison with what there would be in case of no rotation of the earth on its axis and no easterly relative motions. But in the interchange of atmosphere between the two hemispheres, arising from the temperature of the one being greater than that of the other, in which case there is a flow of atmosphere from the colder to the warmer across the equator in the lower strata, and the reverse in the upper strata of the atmosphere, there is no such check to this interchange at and near the equator where the velocity of interchange is greatest, since there are no forces there, as we have seen, to give rise to east or west motions, and if there were such, there would be no forces arising from them acting in the direction of the meridian, since at and very near the equator all deflecting forces arising from the earth's rotation vanish. The interchange, therefore, over the equator, arising from the differences of the temperatures of the two hemispheres in January or July, takes place with very nearly the same facility as they would in case of no rotation of the earth on its axis.

169. Although in January and July the interchange of air arising from differences of temperature does not wholly take place between the equator and the pole, but a part crosses the equator in one direction in the lower, and the contrary in the upper, strata, and makes a circuit in the opposite hemisphere, yet it does not so change the conditions as to give an effect sensibly different from what would arise if the whole interchange were between the equator and the pole simply, as in the case of the mean annual temperature of the earth, provided we use for the value of A_2 in (58) and (59) the half difference between the temperature of the equator and the pole at the given season, instead of for the mean of the year.

The difference between the temperature of the equator and the north pole in winter may be assumed to be, according to the table of § 116, approximately equal 60° and $A_2=-30°$ in winter, and 30° and $A_2=-15°$ in summer. Hence by (59) the values of K in table, § 161, are about twice as great in winter as in summer in the northern hemisphere, and the values also of v' are greater, but not in so great a ratio. In general the values of v, therefore, are about twice as great in winter as in summer, being greater than those of the preceding table for the mean temperature of the year in winter and less in summer. On account of the small differences of temperature between winter and summer in the southern hemisphere the corresponding changes in the easterly velocities of the upper currents there are very small in comparison with those of the northern hemisphere.

Examples.

1. With the value of the January gradient in Table 1, on the parallel of 40° N., and the value of $A_2=-30°$, putting $\tau=0$ and $i=0$, what is the value of v at the altitude of 6 kilometers? (57) and (58).

2. What at the same altitude in summer with $A_2=-15°$, and putting $\tau=20°$?

Oscillation of calm-belts.

170. For the seasons of the year in which the temperatures of the two hemispheres are equal we have interchanging motions between the equatorial and polar regions only, and none between the hemispheres. The positions of the normal calm-belts then are those of their mean position as laid down in Fig. 7. Let the arrows in Fig. 8 represent the velocity of this flow on both sides toward some parallel c, near the equator, the velocities gradually becoming smaller as the air approaches this parallel, as represented by the different lengths of the arrows, and forming a calm-belt at c. But in the summer season of the northern hemisphere, according to what has been stated, there is a flow of air at the earth's surface from the southern to the northern hemisphere across the equator. Let the upper arrows in the figure represent the velocity of this current, which will be sensibly the same for all latitudes near the

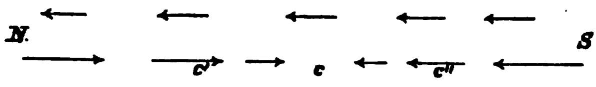


Fig. 8.

equator. The calm-belt then will not be at *c*, but at *c'*, where the velocity of the current north is exactly equal to that toward the south on the north side of *c*. Hence the position of the calm-belt is moved north from *c* to *c'* by this current across the equator from south to north. It will be readily understood that if the current across the equator represented by the upper arrows was reversed, as it is in the winter season of the northern hemisphere, that the position of the calm-belt would be moved from *c* to *c''*. In the annual oscillation of this calm-belt there are corresponding changes in the directions of the wind within the range of oscillation. When the two hemispheres are equally heated, the current represented by the upper arrows vanishes and the wind blows in from the north and the south toward *c*, where there is a calm. In midsummer of the northern hemisphere the current represented by the upper arrows is superadded to the others and we have the resultant of the two, which throws the calm-belt at *c'*, while at *c*, where the calm-belt was, there is a current equal to that represented by the arrows above, and at *c''* there is one equal to the resultant of the two. At the opposite seasons of the year the current at *c* from the north is that indicated by the upper arrows reversed, and at *c'* it is equal to the resultant of the two at that place. At *c*, then, the mean position of the belt, and for some distance on each side, according to the width of the belt, there is a semi-annual change of the direction of the wind from northeast to southeast and the reverse.

The range of oscillation of the central part of the equatorial calm-belt is on the average about 5°. The greatest part of this is on the north side, the other or equatorial side remaining from 2° to 5° north of the equator, with little change between winter and summer. Hence the greater breadth of the belt in summer than in winter.

The same reasoning is applicable to the tropical calm-belts, though not with quite so much force, since the interchanging motions between the hemispheres, which at the earth's surface are represented by the upper arrows in Fig. 8, and which are reversed semi-annually, are not nearly so strong at some distance from the equator, for reasons already given. They are, however, sufficient to cause a considerable oscillation in the positions of these belts with the change of the seasons in the same manner as in the case of the equatorial calm-belt, so that the whole system of these belts is moved north during the spring of the northern hemisphere, and the reverse in the fall.

As in the case of the mean positions of these belts, §166, so in their annual oscillations, the regularity is interfered with by the various abnormal disturbances, and is observed mostly on the oceans only.

171. The permanent abnormal disturbances of the general motions of the atmosphere, which are independent of abnormal temperature disturbances, arise from the deflections of the normal currents by the continents, and especially the mountain ranges. Within the tropics, except at the equatorial calm-belt, we have seen, there is a westerly current at the surface of the oceans and up to an altitude of about a mile. Where this strikes the continent of America it is in some small measure deflected, on the one hand, by the high lands of Mexico, around to the right over the Gulf of Mexico and the southern part of the United States, up into the higher latitudes, where it joins the normal easterly current of those latitudes; and on the other hand, by the continent of South America, and especially the high ranges of the Andes, around to the left, into the higher latitudes of the southern hemisphere where it joins the strong easterly current of those latitudes. Where, also, the easterly current across the North Atlantic strikes the shores of Europe and the interior mountain ranges, it is deflected a little both to the right and the left; to the right along the coast of Spain and Portugal and western coast of North Africa, including the Azores and Canary Islands, until it joins and forms a part of the northeast trades; and to the left by Great Britain and the coast of Norway around by Greenland and Labrador until it joins again the easterly current in more southern latitudes. Hence there are in the North Atlantic, and extending over the contiguous parts of the continent, two large and very gentle whirls or gyrations, the one with its central part in the middle of the Atlantic, or about the parallel of 35°, and the other with its central part in the northern part of the North Atlantic.

The former of these gyrating with the hands of a watch, and the deflective force depending upon the earth's rotation being to the right in the northern hemisphere, there is a tendency to press the air in from all sides toward the central part and to cause an accumulation of air there and the increased pressure which is observed in that region above that on the same parallels on either side. The latter gyrating the contrary way, the tendency is to drive the air away from the central part and cause a diminished pressure there. Only a very small part, however, of the diminution of pressure observed here is due to this cause.

In consequence of the gyration in the southern part of the North Atlantic the northeast trades become more and more west as you approach the continent of America, and finally incline around to the northwest and north and give rise to the prevailing south winds in the southern part of the United States and the Mississippi Valley, especially in summer. For the same reason the prevailing winds on the other side of this gyration, adjacent to Spain and Portugal and the northwest coast of Africa, are first from the northwest, then from the north, until they combine with the northeast trade-winds, and hence these winds near to the African coast have only a small westerly component. The gyration in the northern part of the North Atlantic, arising in this way, causes in part the prevailing southwest winds in the ocean adjacent to Great Britain and Norway and the northeast winds of Greenland.

As these gyrations are caused by the deflections of continents and mountain ranges, they are confined mostly to the lower strata of the atmosphere, and affect only in a small degree by contact and friction the upper strata, where a comparatively strong easterly motion prevails. Hence in the Mississippi Valley the normal surface winds are southerly ones, while at the higher altitudes, and on the higher levels of the slopes and plateaus toward the chain of the Rocky Mountains, westerly winds prevail, in accordance with the normal general motions of the atmosphere in the middle latitudes at these altitudes.

A similar gyration is caused in the same manner in the northern part of the South Atlantic, from which likewise arises an area of high pressure there and the prevailing northeast and north winds in the ocean adjacent to Brazil, and the southwesterly and southerly winds on the other side adjacent to Africa. The whirl, however, in the *southern* part of the South Atlantic is wanting, since Africa does not extend far enough south to deflect the strong easterly current of the higher latitudes, except a small part northward toward the Gulf of Guinea.

Indications of these gyrations are clearly seen on the charts of the winds of the globe,[2] and also on those of M. Brault, for the North and South Atlantic Oceans, based upon many thousands of observations.

On the west side of the North Pacific the trade-winds are deflected in some manner by the East India Islands and continent of Asia, but mostly by the high ranges of mountains of India and China, around, first to the northwest and then to the north, until they join the easterly current of the higher latitudes. This is likewise deflected southward and northward by the west coast of North America, the branch turned toward the south, giving rise to the northwest and north winds adjacent to and on the coast of Southern California and of Mexico, until it comes around and forms a part of the northeast trade-winds. These, on that account, extend here to higher latitudes and come from a point more towards the north than at other places, just as in the case of the Atlantic trade-winds near the northwest coast of Africa. The deflecting force of the currents on the east and west sides diminish a little the atmospheric pressure adjacent to the continents and increase it a little in the central part above what it otherwise would be.

In the South Indian Ocean there is a similar gyration caused by the high lands and mountain ranges of the east coast of Africa on the one hand and Australia and the East India Islands on the other, by which the atmospheric pressure with its maximum near the parallel of 30° is increased a little in the central part and diminished a little on each side next to Africa and Australia.

These gyrations, together with many other irregularities to be considered further on, break up the continuity of the tropical calm-belts around the globe, so that they are observed mostly on the oceans only, and even there they do not extend from coast to coast.

Rain and cloud-belt and dry zones.

172. The northeast and southeast trade-winds passing over an extent of the earth's surface of more than a thousand miles come in from both sides into the equatorial calm-belt, nearly saturated with aqueous vapor. This is especially the case on the oceans where the surface currents move horizontally without any upward deflections, and the same stratum of atmosphere for the most part receives the evaporation of the ocean surface. They would no doubt be entirely saturated if they did not continually pass from cooler to warmer latitudes, whereby their capacity for moisture is being gradually increased. Before arriving at the calm-belt they are gradually deflected upward, and in the central part of the belt, there is necessarily a vertical ascending current, for all the air which comes in from both sides must ascend and return toward the poles as a counter current.

We have seen, § 34, that an ascending current gradually becomes cooler from expansion, at first, while unsaturated, at the rate of one degree nearly for each hundred meters of ascent, and then at rates differing with temperature and altitude as given in Table XIII. If the air arrives at the calm-belt with a dew-point, say 6° below the air temperature, then it ascends 600 meters before it is cooled down to the dew-point, at which condensation and cloud-formation take place, and in this case the lower side of the cloud ring would be at the height of 600 meters. If the temperature of the air at the earth's surface before ascent were 26°, then at the lower side of the cloud it would be 20°. After this, in its further ascent, the rate of decrease of temperature would be by the table 0.42 for each 100 meters without much change at greater altitudes. At the height of 5,000 meters (3.1 miles) the temperature, as determined by the table in the usual way, would be 0° very nearly. By the latter part of Table XIII the weight of aqueous vapor in a kilogram of air at the base of the cloud, temperature 20° and altitude 600 meters, would be 16 grams. But at the altitude of 5,000 meters and temperature 0°, it is 7 grams. Hence a kilogram of air in ascending up to the altitude of 5,000 meters has been cooled from 20° to 0° and 9 grams of aqueous vapor have been condensed and has fallen as rain. In any incipient ascent of vapor, the first vapor condensed is supported by the ascending current in the form of cloud and small particles of rain, and the more the stronger the current is; but only a certain amount can be so retained, and after this is supplied, the amount of the rainfall must be equal to the condensation. The small particles of condensed vapor are carried upward by the current, and the smaller the more rapidly, and the different sizes of small drops being carried up at different rates, they come in contact and combine until they form drops too large to be supported, and they then fall as rain. It is in this way that rain is produced under all circumstances, and when there are no ascending currents there is no rain, though there are often fog and mist. By the old theory of rain, that first advocated by Dr. Hutton, namely, that rain is produced by the meeting of currents of air of different temperatures and capacities for moisture, nothing more than a thin stratum of cloud could be produced between the two currents, for cold and warm currents do not mix, and the amount of vapor condensed would be too small to fall as rain and would be evaporated again before reaching the earth. This was first shown by Espy.[5]

As there is a constant pouring in of air nearly saturated with vapor from both sides into the calm-belt, and an ascent of it there, there is almost a continuous and very abundant fall of rain every day. The daily evaporation within the tropics is about one-fourth inch per day. Within the trade-wind zones, except very near the calm-belt, no rain falls. If, then, all this amount of vaper over belts, say 1,200 miles wide on each side, is carried into the calm-belt 400 miles wide, and is there condensed and falls as rain, then the daily rainfall is 1.5 inches per day, or about 45 feet per year. But as the cloud and rain belt, as the calm-belt, oscillates through a range more than twice as great as its width, this amount is distributed over a zone more than twice as wide; and hence in general much less than half this amount falls at any one place. In Sumatra and Java, however, on and near the equator, a depth of more than 200 inches falls annually at some stations.[6]

173. The oscillations of the rain-belt usually give rise to one rainy and one dry season during the year, but in places, to two rainy and two dry seasons. If a place is situated just within the inner edge of the rain-belt when it has either of its extreme positions, as at *e* or *e'*, it is within the rain-belt about half the year, more or less, according to the width of the belt and range of oscillation, and during the other half without. During the former period it rains almost continuously, while during the other no rain at all falls. This seems to be the position of the Panama Interoceanic Ship Canal, latitude 8.5° north. "The year is divided, as in other parts of the American isthmus and of Central America, into two seasons, the rainy and the dry, the former beginning in the latter part of May and lasting until November, when it gives place to the latter, which lasts until May comes around again. The annual rainfall is from 90 to 140 inches."[7]

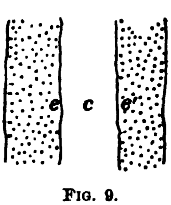


FIG. 9.

If a place is situated nearer the middle of the rain-belt in its extreme position or still further from the center of oscillation *c*, Fig. 9, then there is a long dry and only a comparatively short rainy season.

Where a place is situated on or near the middle of the rain-belt in its central position, as *c*, Fig. 9, then it is readily seen that there must be two wet and two dry seasons during the year, the former occurring during the spring and fall of the northern hemisphere as the rain-belt is moving from its extreme southern to its extreme northern position and back again, and the dry seasons during the times of its extreme positions. This seems to be the position of the Napipi River in the United States of Colombia, about 5° north. "As a rule, two well-marked dry seasons are experienced here with corresponding periods of rain. January, February, and March are the months which constitute the driest and pleasantest season. In April the rains commence, and in May and June they are very heavy. In July a second dry season begins to set in and August and September are generally pleasant and comparatively dry. In October rains again commence, and in November and December they are heaviest."[8] The position of this place is about the middle of the rain-belt in its central position.

On the upper part of the Nile, near the equator, there seems to be the same alternating of wet and dry seasons arising from the oscillations of the rain-belt. The mean position of this belt here, as in most other places, is about 5° N., but the range of oscillation seems to be a little greater. Emin Effendi, in his travels in Equatorial Africa, during the month of December experienced rain at Mreili, 1° 37′ N., every day, and frequently three or four times a day. From Mreili to Rubahga, a little north of the equator, rain fell nearly every day, in great abundance, and the whole country was flooded with water, and the travelers had to wade through water often from two to three feet deep.[9] In Abyssinia, 11° N., on the other hand, the rainy season commences about the middle of May, and continues during the summer months, but during the rest of the year there is no rain. This is because it is within the limits of the belt in its extreme northern position. In June, the Atbara and the Blue Nile pour the whole drainage of Abyssinia into the main Nile, also very much swollen at this time by the floods of the White Nile, and this causes the annual inundation in lower Egypt.

The effects of the rain-belt in Equatorial Africa, Central America, and the East India Islands may be seen on Professor Loomis's Chart of Mean Annual Rain-fall.[6] The zone of 75 inches and over, about 12° in width on the average, and mostly north of the equator, extends entirely across the continents. The islands, also, of Sumatra, Borneo, Java, and others, where observations have been made, are included within this zone; also Central and the northern part of South America. At the base of the Andes, on the headwaters of the Amazon and its tributaries, as likewise in many other parts of the world, there is also an abundant rainfall due to other causes.

174. There are two comparatively dry zones, extending all around the globe, where not broken up by strong abnormal disturbances, comprising the tropical calm-belts and the whole zones of the trade-winds. Under the high pressure of the tropical calm-belts, we have seen, there is a very gradual settling down of the air to supply the outward flow on each side at the earth's surface from beneath these high

pressures. There is here, then, no normal ascending current, as on the rain-belt, but the contrary, and consequently there is no rainfall except when there are ascending currents arising from some abnormal disturbance, and hence the annual rainfall is in general small. Over the trade-wind zones the currents of the lower strata of the atmosphere, which take up and bear away the surface evaporation into the calm-belt, do not allow it to ascend to regions where it can be condensed by the cold of expansion until it arrives at, and ascends in, this belt. Hence little rain falls within these zones where nothing interferes with the regularity of the trade-winds.

These dry zones are observable mostly on the oceans, where the abnormal disturbances which interfere with the regularity of the trade-wind system are comparatively few, except near the continents. But from an inspection of Loomis's chart it is seen that there is at least a strong tendency toward this same condition on the continents of both hemispheres. In the northern hemisphere, in a zone between the parallels of 15° and 40°, are the dry regions of California, Arizona, and Colorado in North America, the great Sahara and Nubian deserts of North Africa, and the dry region of Arabia and Persia in Asia. In the southern hemisphere within the same parallels are the dry regions of the Argentine Republic and of Eastern Patagonia in South America, a large dry region in South Africa, and one comprising the whole of the interior of Australia.

There are, however, certain parts of these zones where there is an abundance of rain. The gyrations on the oceans in both hemispheres, which have been described and explained, § 171, and which interfere with the regularity of the tropical calm-belts, also break up the continuity of the dry zones around the globe. By this means the vapor of the Caribbean Sea and Gulf of Mexico is carried around over Florida and other Southern States, causing a heavier rainfall there than in any other part of the United States, and for the same reason there is an unusually large rainfall in the southeastern part of China, though both these regions are within the comparatively dry zone, taken all around the earth.

The rainfall over the whole of the United States, as far west as the meridian of 100° W., is dependent mostly upon the vapor transported from the Gulf of Mexico and the sea by means of this gyration, for very little comes across the Rocky Mountains from the Pacific Ocean. The vapor is supplied in a similar manner for the rainfall over all China.

Local large and scanty annual rainfalls.

175. There are several conditions which are all more or less favorable to large rainfalls in connection with the general motions of the atmosphere. These are: (1) That the rain-bearing winds shall blow from the ocean, and best, from warmer to cooler latitudes; (2) That there shall be ranges of high mountains to deflect the currents upward, which

are nearly or quite normal to the direction of the wind; (3) That the place shall be near the ocean.

If the winds come from the ocean, and especially from a lower to a higher latitude, they are more apt to be nearly saturated with aqueous vapor. If these strike against ranges of high mountains, especially if they are coast ranges, the nearly saturated air currents are deflected upward and the vapor is rapidly condensed and falls in rain, upon the same principles as in the equatorial rain-belt. In this manner the easterly winds of the trade-wind zones, charged with vapor from having passed over a long distance of water, are thrown upwards by the steep and lofty range of the Andes, and this results in the heavy annual rainfall on the eastern side of the headwaters of the Amazon and its tributaries, as represented on Loomis's chart. For the same reason the zone of heavy precipitation in the central part of Africa is wider on the eastern side of the continent, and the rainfall rendered more abundant.

In the higher latitudes of the northern hemisphere the prevailing west and southwest winds, blowing from the Pacific and Atlantic Oceans and striking nearly perpendicularly the line of coast ranges of mountains, the vapor of the currents deflected upward is condensed and gives rise to the large rainfall of Oregon and Washington Territory on the northwest coast of America on the one hand, and along the coast of Norway on the other. On the west coast of the southern part of South America there is also a very rainy region. The west winds of the South Pacific Ocean, striking the west side of the Andes, are deflected nearly vertically upward, and consequently there is a very large rainfall. But the southern part of Africa does not extend far enough south into the parallels of the westerly winds of the lower strata of the atmosphere for a similar effect to be produced there, even if there was as high a mountain range as on the southern part of South America.

As the rainfalls on land are dependent in a great measure upon the aqueous vapor supplied by the ocean, as a rule much more rain falls on and near the sea-coasts than in the interior of the continents. And this is especially the case if the vapor-bearing winds have to pass over coast ranges of mountains in passing into the interior. For in passing over the range most of the air has ascended to an altitude considerably greater than that of the mountain, where it has become so much cooled and its capacity for moisture so much diminished that there is but little vapor remaining, and in passing on over the dry interior it receives very little more from evaporation. The interiors of large continents, especially in the higher latitudes, are therefore comparatively dry and the annual rainfalls light. It is seen from Loomis's chart that in the interior of North America east of the Rocky Mountains, especially up in the cooler and drier latitudes, there is a large area in which the annual rainfall is very scant. The same is also seen in Tartary and Mongolia, in the interior of Asia.

CHAPTER IV.

CYCLONES.

I.—THE FUNDAMENTAL EQUATIONS.

Temperature conditions.

176. In the theory of the general motions of the atmosphere, treated in the preceding chapter, the only disturbing cause taken into account is the difference between the normal temperatures of latitude between the equatorial and polar regions of the globe, neglecting all local permanent variations of temperature in the different longitudes on the same parallel, and also all variations of temperature of a temporary and transient character. The former more general variation of temperature, we have seen, gives rise to two similar systems of atmospheric circulation, each one having one of the poles of the earth for its center, and embracing a whole hemisphere. In the more local and temporary disturbances in case of cyclones the causes are similar, but the areas of disturbance embrace only a small part of a hemisphere.

177. In the unequal distribution of temperature over the earth's surface, arising from various causes, it must frequently happen that, besides the great normal inequality of temperature between the equatorial and the polar regions, there are also more local, abnormal irregularities, either temporary or permanent, independent of latitude. If it should happen, as it must frequently, that these local areas of abnormal temperature are of a somewhat circular form, with a gradient of temperature increasing or decreasing with more or less regularity from the central to the exterior part, they furnish conditions somewhat similar to those of the general motions of the atmosphere, in which there is a gradient of increasing temperature from the poles to the equator. These local temperature disturbances give rise to interchanging motions between the central and exterior part of the disturbed area, and these in turn, on account of the deflecting force due to the earth's rotation, give rise to gyrations around the center, often of great violence. Local disturbances of this character are called cyclones.

178. In the general motions of the atmosphere the temperature inequality between the equatorial and polar regions is maintained by a constant difference in the vertical intensity of the solar radiation upon the two regions, and hence the general motions of the atmosphere are unceasing. In the case of cyclones a mere initial difference of temperature between the central and exterior parts can only give an initial

start to the system of motions, and a permanent cyclone cannot be maintained, unless this difference of temperature is kept up from some constant cause, for the interchanging motion between the central and exterior parts would soon equalize the temperature and destroy the temperature gradient upon which the motions depend.

There is a certain state of the atmosphere, however, with which, from an initial start arising from temporary differences of temperature, the cyclonic action may be continued for some time, and that is the state of unstable equilibrium explained in § 38. In this case if the central region is warmer than the surrounding parts so as to give rise to incipient interchanging motions *toward* the central part below and *from* it above, with ascending currents in the interior, and if the air in ascending here does not cool with increase of altitude at a rate which is less than the rate by which the temperature diminishes with increase of altitude in the comparatively quiet surrounding atmosphere, then the temperature in the interior must continue to be greater and the density less than in the surrounding part until such an inversion of the whole atmosphere in the vicinity takes place and the vertical temperature gradient becomes so changed that the condition of unstable equilibrium is broken up. On the other hand, if the vertical temperature gradient in the surrounding atmosphere is less than the rate with which ascending air is cooled in the interior part, then it is readily seen that, even with an initial higher temperature in the interior, this part, taken in all its extent through the upper strata, must become cooler than the surrounding parts; for as the air at the bottom is supplied by the currents coming in from all sides, its temperature must soon become the same as that of the surrounding parts, but on account of the difference in the ratio of cooling with increase of altitude the temperature of all the strata above the earth's surface must be cooler and more dense and the ascending currents and all initial cyclonic action must cease.

The conditions of a cyclone, therefore, depend very much upon the rate of decrease of temperature with increase of altitude, that is, upon the vertical temperature gradient; since without the state of unstable equilibrium, which depends upon this rate, any initial horizontal temperature gradient is soon destroyed.

In the general motions of the atmosphere this latter condition is not required, since the solar radiation is so great a disturber of the equality of temperature between the equatorial and polar regions that the difference is only very slightly affected by any difference in the vertical temperature gradients in different latitudes arising from ascending and descending currents. If in a cyclone there is likewise some permanent cause of temperature disturbance, some source from which the central part is continually receiving more heat than the surrounding parts, so as to keep up a difference of temperature too great to be reversed by the changes arising from ascending and descending currents with different vertical temperature gradients, then in a cyclone, as in the general motions of the atmosphere, the condition of unstable equilibrium is not required; but even in this case this condition increases the energy of the cyclone.

179. In the case of dry or unsaturated air, § 36, the rate of decrease of temperature of an ascending current of air is 1° nearly for each 100 meters of ascent, and hence if in the surrounding comparatively undisturbed part the vertical temperature gradient is greater than this, we have the state of unstable equilibrium. But so great a vertical gradient of diminishing temperature is by no means a normal one, or even one which frequently occurs, but is observed only under certain rare circumstances and in the lower strata of the atmosphere very near the earth's surface. Hence we never have the conditions of a cyclone of long continuance in very dry air; for although some local part of the atmosphere may happen to be warmer in the central than in the surrounding parts, yet this can only give rise to an initial interchanging motion between the central and surrounding part before the temperatures of the two become equalized by the rapid cooling of the ascending air in the central part, when this motion must cease.

In saturated air, however, we have the unstable state with a vertical gradient of decreasing temperature which is much less than 1° for 100 meters. The rate of cooling in an ascending current of air in this case is that of the Table XIII, which, it is seen, varies with temperature and altitude, the rate per 100 meters for medium temperatures and altitudes being less than half of what it is in all cases in dry air. This rate is about the same as that of the normal or average vertical gradient in the atmosphere. Hence where the air is saturated or nearly so, we frequently have the state of unstable equilibrium, which, we have seen, is a very important condition of a cyclone.

If the air is not completely saturated at the earth's surface, and we have the initial or primary cause of a cyclone, namely, a greater temperature in the central than the exterior part of some local temperature disturbance, to cause an initial ascending current there, the decrease of temperature for each 100 meters of ascent is 1° until the air has ascended as many hundred meters as there are degrees of difference between the air and the dew-point temperatures, after which the rate of decrease is that given in Table XIII. The complete temperature conditions of a cyclone, therefore, rarely extend down to the earth's surface, but the interchanging and gyratory motions, commencing first up in the cloud regions, are soon propagated downwards to the earth's surface by the action through friction of the upper strata upon the lower ones. The pressure, also, on the lower strata being diminished by the rarefaction of the air above in the central part of the cyclone, the air of these lower strata tends to rise up in the central part and then that of the surrounding parts flows in to take its place, and in so doing it is necessarily drawn also into a gyration around the center.

180. The aqueous vapor of the atmosphere is a very important, though not an indispensable, condition of a cyclone. It is important because, where the air is saturated with it, we have the unstable state which is so essential to continued cyclonic action, with a vertical gradient of decreasing temperature which is much less than in the case of dry air, and one which frequently occurs in nature. If the vertical gradient of decreasing temperature were greater than 1° for 100 meters, we should have the unstable state, and, with any slight initial cause of disturbance, the complete conditions of a cyclone in perfectly dry air. Of course the tendency of the ascending and descending currents of such disturbances, since in both the rate is 1° nearly for 100 meters, the former a decreasing and the latter an increasing one, is to bring the temperature gradients in the end to the same, as it is in the case of saturated air to bring the vertical gradient to a normal or average one.

181. Where the initial state of the air is such that the central part is colder than the surrounding air, it gives rise to an interchanging motion toward the center above, a very gentle downward motion in the interior, and a motion from the center below. If the temperature conditions with regard to the vertical gradient were such as belong to the unstable state of dry air, the motions would continue until this state were changed by such motions; and if the initial state of the air were that of complete saturation, there would be little advantage from it, since saturated air in a descending current becomes at once unsaturated, and the effect of the rising of as much air in the surrounding parts, since it would extend over a so much greater area, would be very small in raising the surrounding temperature.

Unless for some reason the central part of some area of disturbed temperature is kept colder than the surrounding part, we cannot have the conditions of a cyclone with a cold center which will continue long unless the atmosphere is in a state of unstable equibrium for dry air, which never occurs except near the earth's surface.

182. Perfectly regular conditions of a cyclone would comprise a circular area of no very great extent, with a regular gradient of either increasing or decreasing temperature from the center to the circumference, as is the case very nearly in the general motions of the atmosphere where we consider only the normal temperatures of latitude. Of course these regular conditions never occur in nature, and generally only very rough approximations to them, but the best that can be done in the theoretical consideration of cyclones is to assume such regular conditions, and to form equations representing, as nearly as possible, such conditions. The equations even for such conditions, as in the case of the general motions of the atmosphere, can only be imperfectly and approximately solved; but still results may be deduced from such a solution very important in enabling us to understand the various phenomena connected with these motions, even where they arise from conditions differing much from the regular conditions assumed.

Gyratory motion of a small portion of the earth's surface.

183. A comparatively small portion of the earth's surface around the north pole, with the pole in the center, turns with reference to space around that pole like a disk with an angular gyratory velocity, equal to that of the earth around its axis, and in a manner contrary to that of the hands of a watch, or, as is usually said, from right to left. The same is true with regard to a similar area around the south pole, except that it would seem to one at the south pole to turn the contrary way, that is, *with* the hands of a watch. At the equator such an area would have a progressive circular motion eastward around the earth's axis, but none of gyration around its center. From the poles to the equator, therefore, it is reasonable to suppose that the motion is partly progressive and partly gyratory, and that the gyratory motion around the center is some function of the polar distance which decreases with increase of polar distances and vanishes at the equator.

Let Fig. 1 and Fig. 2 represent two intersecting planes passing

Fig. 1.

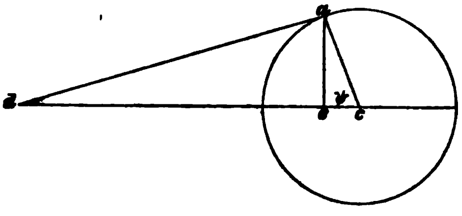
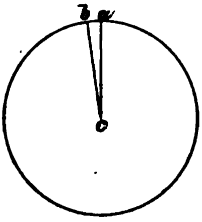


Fig. 2.



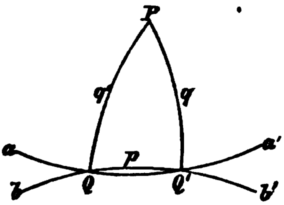
through the point a on the earth, of which the polar distance is ψ, the first passing through the axis around which the earth revolves with a gyratory velocity n, and the other perpendicular to it and intersecting it at e. If ab, Fig. 2, represents a very small instantaneous motion of which the gyratory velocity around the axis is n, and n' represents the corresponding gyratory motion around the point d in a tangent plane touching the earth's surface at the point a, and intersecting the earth's axis produced at d, then we shall have, since in the one case ae and in the other ad is the distance of the common motion ab from the center,

$$\frac{n'}{n}=\frac{ae}{ad}=\frac{\sin\psi}{\tan\psi}=\cos\psi$$

And hence

$$n'=n\cos\psi \tag{1}$$

Every part of a plane gyrating as a disk around a center has an instantaneous motion which is composed of a progressive motion perpendicular to the radius and of a gyratory motion with a velocity equal to that of the gyrating plane. Hence the instantaneous motion of a part of the earth's surface at a, so small that it may be regarded as coinciding with a tangent plane at a, is composed of a progressive motion ab, and a gyratory motion of which the velocity is $n \cos \psi$, or the velocity of the earth's rotation multiplied into the sine of the latitude.



184. The preceding relation may be also demonstrated in the following manner: Let P, Q, and Q' be the three angles of an isosceles spherical triangle $P\ Q\ Q'$, P coinciding with the pole of the earth, and p, q, and q' the opposite sides, respectively. We then have, from the well-known trigonometrical relations of a spherical triangle,

$$\tan \tfrac{1}{2}(Q+Q')=\frac{\cos \frac{1}{2}(q-q')}{\cos \frac{1}{2}(q+q')} \cot \tfrac{1}{2}P$$

Let the side p of the triangle be produced to b and b', and let $aQQ'a'$ be the parallel of latitude passing through Q and Q'. The change in the direction of this parallel, which is the line of motion, between Q and Q', is twice the angle $aQb=a'Q'b'$, and these are the complements of the angles Q and Q'. But since $q=q'=\psi$ of the preceding notation, and consequently $Q=Q'$, the preceding relation becomes, since $\tan \frac{1}{2}(Q+Q')=\tan Q=\tan (90^\circ-aQb)=\cot aQb$,

$$\cot aQb=\frac{1}{\cos \psi} \cot \tfrac{1}{2}P$$

or

$$\tan aQb=\cos \psi \tan \tfrac{1}{2}P$$

Now, if we suppose the angle at P to be infinitely small so as to represent an instantaneous change in the angular motion of rotation, then the angle aQb likewise becomes infinitely small, and we have

$$d(aQb)=\tfrac{1}{2} \cos \psi dP$$

Or, dividing both members by the element of time dt, we get, since $2D_t(aQb)$ is the velocity of change of direction of motion with regard to space, and D_tP is the angular velocity of rotation,

$$n'=n \cos \psi$$

the same as in (1).

Equations deduced from those of the general motions of the atmosphere.

185. If the general equations (9), § 143, were applied to a portion of the earth's surface around the pole so small that it might be regarded as coinciding with the tangent plane at the pole, then θ would be so small for any part of this area that we could put $\cos \theta=1$. For a similar portion of the earth's surface around any point a, Fig. 1, we should have similar conditions, since progressive motions with reference to space do not enter into them, but only the gyratory motions. Instead, however, of a gyratory velocity n, we should have in this case one equal to $n \cos \psi$.

If now we adopt the same notation as in § 143 in the case of the general motions of the atmosphere, except that U and V represent linear co-ordinates in directions of a radius drawn from the center of the area under consideration and at right angles to this radius respectively, and u and v the corresponding velocities of relative motion in these directions, we thus get for the equations of condition in this case by putting in (9), § 143, $\cos\theta=1$, and $n\cos\psi$ for n,

$$(2)\qquad \begin{aligned} -\frac{d\log P}{\alpha dU}&=-(2n\cos\psi+\nu)v+\frac{du}{dt}+F_u \\ -\frac{d\log P}{\alpha dU}&=(2n\cos\psi+\nu)u+\frac{dv}{dt}+F_v \\ -\frac{d\log P}{\alpha dX}&=g+\frac{dx}{dt}+F_x \end{aligned}$$

in which, putting $\rho=$ the linear distance from the center, we have sensibly where ρ is not very large

$$(3)\qquad \rho=r\sin\theta \qquad\qquad \nu=\frac{v}{\rho}=\frac{v}{r\sin\theta}$$

Where the vertical motions are so small, as they are mostly, that they can be neglected, we get from (11), § 144, by making the same changes

$$(4)\qquad \begin{aligned} -\frac{d\log P'}{\alpha dU}&=-(2n\cos\psi+\nu)v+\frac{du}{dt}+F_u-gh\frac{d\log\alpha}{dU} \\ -\frac{d\log P'}{\alpha dV}&=(2n\cos\psi+\nu)u+\frac{dv}{dt}+F_v-gh\frac{d\log\alpha}{dV} \end{aligned}$$

as the equations of horizontal motions of the atmosphere.

In the case of the regular conditions of a cyclone as assumed above, α, which is a function of τ, becomes independent of V, and hence the last term of (4_2) vanishes, and also the first member, unless, in the case of no friction, it depends upon initial motions. But in the case of friction such initial motions soon disappear, and then the first member in (4_2) vanishes.

In the case of cyclones, as in that of the general motions of the atmosphere, § 144, it is necessary to have an expression of the temperature in terms of the distance ρ from the center. In the case of the general motions of the atmosphere this is approximately known from observation, and is of the form of (12), §144. In the case of cyclones this is not known, and it is therefore necessary to assume some expression of the temperature which we may suppose to represent it approximately. This must be of such a form as to make the temperature a maximum or minimum in the center and to increase or decrease from the center to the exterior limit according to some probable and regular gradient. These conditions are satisfied by an expression of the following form:

$$(5)\qquad \tau=A_0+A_1\cos 2\varphi$$

in which, putting ρ_0 for the value of ρ at the exterior limit,

$$(6)\qquad \varphi=\frac{\rho}{2\rho_0}\pi$$

and in which A_0 is the value of τ at the mean distance between the center and outward limit. From these we get, since $dU = d\rho$, both being linear distance from the center,

$$\frac{d\tau}{dU} = -\frac{\pi}{\rho_0} A_1 \sin 2\varphi$$

But from (4), § 142, we get, as in § 144,

$$\frac{d \log \alpha}{dU} = -\frac{.004}{1+.004\tau} \frac{d\tau}{dU}$$

Hence we get

$$\frac{d \log \alpha}{dU} = \frac{.004\pi A_1}{\rho_0(1+.004\tau)} \sin 2\varphi$$

With this value in the last term of (4_1), and putting the first member of (4_2) and the last term of the second member equal 0, for reasons already given, we get

$$(7)\quad \begin{aligned} -\frac{d \log P'}{\alpha dU} &= -(2n \cos \psi + \nu)v + \frac{du}{dt} + F_u - \frac{.004\pi A_1}{\rho_0(1+.004\tau)} gh \sin 2\varphi \\ 0 &= (2n \cos \psi + \nu)u + \frac{dv}{dt} + F_v \end{aligned}$$

In the expression of (5) A_0 is the mean of the temperature between the center and exterior limit. In the last term of (7_1) this mean value of τ can be used without material error, and if this were taken for the altitude of $\frac{1}{2}h$ it would be better than the value at the earth's surface.

These are the equations of the horizontal motions of a regular cyclone with a temperature increasing or decreasing from the center to the exterior according to the assumed expression of (5). If A_1 is positive the central part is the colder and the gradient is one of an increasing temperature from the center out, but the reverse if A_1 is negative, which is the usual case of ordinary cyclones.

All that has been stated in § 145 with reference to equations (13) of that section is applicable to the preceding equations (7).

II.—SOLUTION IN CASE OF NO FRICTION.

1°. *In case of no temperature disturbance.*

186. Equations (2) and (7) differ but little from (9) and (13) in the case of the general motions of the atmosphere. The solutions, therefore, and likewise the results, in the two cases are very similar where they are not precisely the same.

In the most simple case, that of a free body moving over the earth's surface, supposed to be entirely smooth and without friction, we obtain from (7), just as (14) were obtained from (13), § 145, equations which are similar to those of (14), § 145, and which are deducible from them by putting $\cos\theta=1$, $n\cos\psi$ for n. By introducing these changes into (16_1), § 145, we get for the deflecting force arising from the earth's rotation

$$F=2ns\cos\psi$$

which is precisely the same as the former for the middle of the comparatively small tangential area, since ψ is the polar distance of that point. It was assumed, in forming the equations (2) and (7), that the area under consideration coincided sensibly with the earth's surface, and as this is strictly true for the central point only, the result above can be strictly correct for that point only, and for others near to it only approximately so.

187. Where we consider only a circular portion of the atmosphere over a comparatively small area of the earth's surface and suppose it to be moved without friction by forces, however small, acting only in vertical planes passing through the center, we obtain from (7_2), for each particle of air,

$$\rho^2(n\cos\psi+\nu)=c \tag{8}$$

this being the integral of that equation, as may be shown by differentiation. This is the same as (8), § 143, if we suppose that the area is so small that $\sin\theta$ does not differ sensibly from the arc, or $r\sin\theta$ from the linear distance from the center.

By proceeding in the same manner as in § 147 and 148, we obtain, instead of (22), § 148,

$$C=\frac{\int\rho^2(n\cos\psi+\nu')dm}{m} \tag{9}$$

But this is obtained directly from (22) by putting $n\cos\psi$ for n and ρ for $r\sin\theta$ by (3), which are the only differences in the equations in the two cases.

Where the initial state of the atmosphere is that of rest with reference to the earth's surface, we have $\nu'=0$, for each particle dm. The integration of (9), therefore, in this case gives

$$C=\tfrac{1}{2}\rho_0{}^2n\cos\psi$$

putting ρ_0 for the value of ρ at the exterior limit of the area. With this value of C, which is the common value of c, for all the particles after they assume the same angular velocity of gyration at the same distance from the center, substituted in (8) we get, putting for ν its value in (3_2),

$$v=\left(\frac{\rho_0{}^2}{2\rho^2}-1\right)\rho n\cos\psi \tag{10}$$

If in this equation we put $v=0$, we get

$$\rho=\frac{\rho_0}{\sqrt{2}} \tag{11}$$

for the distance from the center at which the gyratory motion vanishes and changes sign. Between this distance and the center the gyrations, by the notation adopted, are positive, that is, from right to left in the northern hemisphere where $\cos\psi$ is positive, but beyond that distance they are the contrary way.

188. By treating equations (2) in precisely the same manner as those of (9) in § 149 in the preceding chapter were, it is readily seen that we must obtain the same results, since the only two terms which differ in the two sets of equations, namely, those depending upon the gyratory velocity $(2n+\nu)$ in the one case and upon $(2n\cos\psi+\nu)$ in the other, vanish in both cases. Hence equations (25), (26), (27), and (27′) in the preceding chapter hold also here in the case of a cyclone without friction.

If in (27′) we suppose P_0 and v_0 to be the values of P and v at the limit of the circular area where $\rho=\rho_0$, we get from (10) $v_0^2=\frac{1}{4}\rho_0^2n^2\cos^2\psi$. With this and the general value of v^2 in (10), (27′) becomes by means of (4) and Table XIV

$$\log P_0-\log P=\frac{14732\times10^{-19}}{1+.004t}\cos^2\psi\left(\frac{\rho_0^4}{4\rho^2}-\frac{5}{4}\rho_0^2+\rho^2\right) \tag{12}$$

in which the expression is adapted to common logarithms.

If P_0 is the value of P where $v=0$, then (27′), § 149, gives by (10)

$$\log P_0-\log P=\frac{14732\times10^{-19}}{1+.004\tau}\cos^2\psi\left(\frac{\rho^4}{4\rho^2}-\rho_0^2-\rho^2\right) \tag{13}$$

From these expressions it is seen that on account of the term containing ρ^2 in the denominator the first members become very large and consequently the value of P very small in comparison with P_0 where ρ is small in comparison with ρ_0.

If we put the differential of (12) equal 0, we get

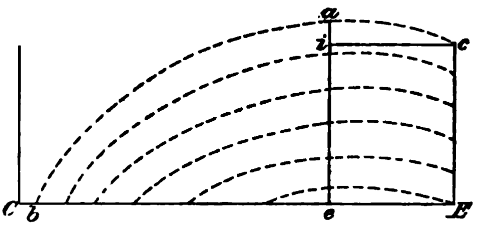
$$\rho=\frac{1}{\sqrt{2}}\rho_0$$

the same as (11), for the value of ρ where the pressure P is a maximum. Hence this occurs, as in the case of the preceding chapter, § 149, where v vanishes and changes sign as any particle passes from the exterior to the interior of the circular part of the atmosphere under consideration or the reverse. This, however, as in the former case, is upon the hypothesis that the initial state of the air is that of rest. With other initial states we get of course very different relations between ρ and ρ_0. For instance, the whole air might have such an initial gyration that there would be no maximum in the final result, and the highest pressure would be still at the exterior limit.

189. Since by (12) P becomes very small near, and vanishes at, the center where $\rho=0$, however great the value of P_0, and also has a max-

imum at the distance from the center determined by (11), which is at a certain distance between the center and the outer limit, but nearer the latter, the figures of the strata of equal pressure must assume forms such that sections by vertical planes are represented by the dotted lines in Fig. 1, which are similar to those of Fig. 3, § 149, except that in this case they belong to only a very small portion of the earth's surface instead of a whole hemisphere as in the former case.

Fig. 1.



Between the center C and e, the distance from the center of greatest pressure, the gyrations in the northern hemisphere where $\cos \psi$ is positive are from right to left, by (10), and very near the center the velocity is very great; but beyond e they are the contrary way, the maximum velocity being by (10) equal $\frac{1}{2} \rho n \cos \psi$. In the southern hemisphere the gyrations of corresponding parts are all reversed.

The value of P with reference to P_0, by (12), and likewise the value of v, by (10), depend upon ψ, the polar distance of the center of the area of disturbance. At the equator, where $\cos \psi = 0$, we have $v = 0$, and $P = P_0$, and hence no gyrations or any disturbance of pressure. But it must be remembered that this is upon the hypothesis of an infinitely small temperature disturbance between the interior and exterior parts, since α in the preceding integrations has been regarded as a constant. For however small this disturbance and the consequent interchanging motions between the interior and exterior parts, the final result is precisely the same, but the approximation to this result is slower and the time in which it is sensibly reached much longer.

Since (29) of the preceding chapter is entirely independent of the earth's rotation or of area of extent, but the relation between s and s_0 depends simply upon difference of elevation of the initial and final velocities, it is also applicable in this case, as likewise (30) and (31). The gyratory velocity v being 0 at the distance e from the center C, Fig. 1, the velocity v at the outer limit E is the same as would be acquired by a body falling through the distance $X - X_0$ in (31) or from a to i' in Fig. 1. Likewise the value of v at b, where the stratum bac comes to the earth's surface, is the velocity acquired by a body in falling through X or from a to e, Fig. 1. This, however, is upon the hypothesis of an infinitely small interchanging motion between the interior and exterior parts, since (30) and (31) have been deduced from (29), § 149, upon this hypothesis. If, however, u and x in (26), which depend for their values upon this interchanging motion, should have considerable values, but small in comparison with v, the result would be very nearly the same. The last paragraph of § 149 is likewise applicable here.

In the preceding results it is assumed either that the interchanging motions depend upon an initial impulse, or else that where there is friction the temperature disturbance is merely sufficient to overcome the friction of small velocities, since they have been deduced upon the hypothesis that α in (9) is a constant, and so independent of temperature variations with regard to U, V, and X.

2°. *Special solution in case of temperature disturbance.*

190. Another solution of (1) in the case of no friction, but with difference of temperature between central and exterior parts, similar to that of § 150 of the preceding chapter, can be obtained by putting $u=0$ in (7_2) and $v=$a constant. This satisfies this equation, and we thus get $D_t u=0$, and $F_u=0$ in (7_1), and it then becomes

$$\frac{d \log P'}{\alpha dU}=(2n \cos \psi+\nu)v+\frac{.004\ \pi A_1}{\rho_0(1+.004\ \tau)}\, gh \sin 2\ \varphi$$

At the earth's surface where $h=0$, we get, since the last term vanishes in this case,

$$\frac{d \log P'}{\alpha dU}=(2n \cos \psi+\nu')v'$$

putting v' and ν' for the values of v and ν at the earth's surface.

Subtracting the latter of these equations from the former, and neglecting the small difference between ν and ν', since it is in general very small in comparison with the whole gyratory velocity $2n \cos \psi+\nu'$ at the surface, we get

$$(14)\qquad \begin{aligned} v'-v&=\frac{.004\ \pi A_1\, gh \sin \varphi}{\rho_0(2n \cos \psi+\nu')\,(1+.004\ \tau)} \\ &=\frac{0.1232\ A_1 \sin^2 \varphi}{(2n \cos \psi+\nu')\,(1+.004\ \tau)}\cdot\frac{h}{\rho_0} \end{aligned}$$

The value of ν' is generally so small in comparison with $(2n \cos \psi+\nu')$ in the middle zone and exterior part of a large cyclone that it may be neglected; (14) then, by Table XIV, becomes

$$(14')\qquad v'-v=\frac{845\ A_1 \sin \varphi}{\cos \psi(1+.004\ \tau)}\cdot\frac{h}{\rho_0}$$

The last two expressions make $(v'-v)$ vanish at the center where, by (6), $\varphi=0$, and at the outer limit where $\varphi=180°$, and a maximum where $\rho=\frac{1}{2}\rho_0$, or where by (6) $\varphi=90°$.

Where A_1 is positive, as it is by (5) where the temperature decreases from the center outward, v is less than v', and hence the gyratory velocity above is less than it is at the earth's surface, and in proportion to the altitude h. This is evident from a more general consideration of the subject, in which, however, no quantitative results are obtainable. The air in the interior of the cyclone is expanded upward and the strata of equal pressure raised. This diminishes the gradient of pressure, decreasing toward the center, and hence a less force arising from the gyratory velocity, and consequently a smaller velocity of this kind is required above to counteract the gradient there than is required below, where it is not diminished by expansion from increased temperature in the interior. Where there is no interchanging motion, as assumed, there must be no residual of force above to move the air out from the interior, nor below to move it in toward the center, and hence the force arising from the gradient and that depending upon the gyratory motion must exactly counteract each other for every part, both above and below.

If A_1 is negative, that is, if by (5) the interior is the colder and there is a gradual increase of temperature outward from the center, then the reverse takes place, and the gyratory velocity is greater above than below, just as in the case of the general motions of the atmosphere without friction in § 150, where the interior or polar regions are the colder and the temperature increases toward the pole. If the assumed gyratory velocity v' at the earth's surface is 0, then, as in the case of the general motions of the atmosphere, the gyrations in the northern hemisphere are still from right to left above and the pressure gradients decline toward the center, as they do toward the pole in the case of the general motions, as is represented in Fig. 4, § 150.

Examples.

1. If $\rho_0=1{,}000^{km}$ (10), what is the gyratory velocity v on the parallel of 50° at the distance from the center of $\rho=100^{km}$?

2. If $P_0=0.76^m$ at the outer limit E, Fig. 1, at the distance of $\rho_0=1{,}000^{km}$, with $\psi=45°$ and $\tau=0°$, what is the pressure P at the distance $\rho=100^{km}$? (12.)

3. If $P_0=0.78^m$ at the point e, Fig. 1, where the pressure is a maximum, and $\rho_0=100$ miles, what is the value of P at the distance $\rho=5$ miles? (13.)

4. If $\rho_0=1{,}000$ miles and $A_1=10°$ F., what is the value of v at the altitude of 3 miles and latitude of 45° and distance $\rho=600$ miles, supposing there is no gyratory motion at the surface of the earth, that is, that v' and ν' equal 0? (6) and (14'). Ans. 34^m.

III.—SOLUTION IN CASE OF FRICTION.

Introduction.

191. In the preceding case friction between the atmosphere and the earth's surface has not been considered, and only an exceedingly small amount of it has been supposed to exist between the different parts, merely sufficient to destroy in time all initial motions not depending upon some regular disturbing cause, and this latter has been supposed to be infinitely small and to be continued only until regular interchanging motions between the interior and the exterior parts of the area of

disturbance have become established. It is now proposed to treat the equations in the case in which the friction, both between the atmosphere and the earth's surface and between the different parts of the atmosphere, is so great as to require a considerable amount of force to overcome the frictional resistances and to keep up the motions when once established. In the preceding case complete solutions of the equations for the most part and quantitative results have been obtained for the regular conditions of § 182, but in this case, even for these regular conditions, as has been stated, the equations can be only very imperfectly solved, and in general no quantitative results be obtained, but nevertheless results which are very important in showing the general tendency of the disturbing forces and in enabling us to explain and even anticipate many of the phenomena which usually occur.

Interchanging motions.

192. In the case of cyclones the difference of temperature between the interior and exterior parts gives rise to an interchanging motion between these parts, just as in the case of the general motions of the atmosphere the difference between the temperature of the polar and equatorial regions of the earth gives rise to such a motion. In the case of the latter, however, the motions are toward the polar or central part in the upper strata and from it in general in the lower ones, but in an ordinary cyclone, in which the interior part is the warmer, these motions are reversed, being mostly toward the center in the lower strata and from it in the upper ones.

The force which overcomes the inertia of the air in the initial motions where the initial state is that of rest, and which afterwards continues to overcome the frictional resistances to these motions, depends upon the temperature gradient expressed by the last term of the second number of (7_1). In the initial state before motion commences we have P' equal a constant and also $F_u=0$, and (7_1) then becomes

$$\frac{du}{dt}+F_u=\frac{.004\pi A_1}{\rho_0(1+.004\tau)}gh\sin 2\varphi \tag{15}$$

in which F_u only acquires a value after motion has commenced. This equation is similar to that of (39) of the preceding chapter, and the effect in first increasing the pressure in the central part and of producing a motion toward the center above and from it below, and of causing a neutral plane in which there is no motion, as explained in § 153, is precisely the same in both cases. The equation of continuity expressed by (40), § 153, must also be satisfied by the interchanging motions in this case.

Gyratory motions.

193. The force which overcomes the inertia and friction of the gyra-

tory motion of any part of the air is obtained from (7_2), which may be put into the following form:

$$(16) \qquad \frac{dv}{dt}+F_v=-(2n\cos\psi+\nu)u$$

in which the last member is the force which overcomes the inertia and friction expressed by the first member, and is similar to (42) in the last chapter.

This force, it is seen, depends upon u and the earth's rotation, so that without a motion to or from the center and without rotation of the earth on its axis, there is no force to give rise to a gyratory motion, for ν only acquires a value after this motion has once set in. Since, also, $\cos\psi$ vanishes on the equator, there can be no gyratory motion there in any case.

When the interchanging motion is established, as explained above, the motion toward the center below, by the principle of § 145, or by (16) above, in which u in this case is negative, tends to give rise to a gyratory velocity v from right to left in the northern hemisphere, but the contrary in the southern, since $\cos\psi$ changes sign there. The motions above *from* the center tend to give rise to gyrations the contrary way, but the tendency of the friction between the strata is to constantly reduce them to the relations of (14) in the case of no friction, in which the gyrations, algebraically considered, are greater below than above in all parts. The difference, however, must be always a little less than that given by (14), since this, having been obtained upon the hypothesis of no interchanging motions or of $u=0$, must be the limit of difference. If the relative values of v' and v, the gyratory velocities below and above, were such as to completely satisfy (14), the force depending upon the temperature gradient and that arising from the centrifugal force of the gyrations and the deflecting force of the earth's rotation, as expressed by the first term of the second member of (7_1), would exactly counteract each other, and the interchanging motion would cease, u in (16) would vanish, and there would then be no force to overcome the friction of the gyrations.

The velocity of gyration of the earth's surface at unit distance around the center of the cyclone being $n\cos\psi$, the tendency of the air of the lower strata in moving toward the center, to run into a gyration around that center relatively to the earth's surface, is similar to that of water in a shallow basin having a very small gyratory motion, to run into a gyration relatively to the basin around the center, if the water is allowed to run out through a hole in the center of the basin, from which arises a motion toward the center. In the case of the air in a cyclone, however, instead of running down, it rises up in the central part and flows out above, and the tendency to gyration from this motion the contrary way tends to counteract the other.

We have seen that in the case of no friction the atmosphere tends to run into a gyration in the one direction in the interior part, and the contrary in the exterior part, and that the velocity of gyration becomes very great near the center. In the case of friction between the air and the earth's surface there is the same tendency, but the weak deflective forces upon which the gyrations depend are sufficient to overcome the frictional resistances of only very small velocities comparatively, and hence they cannot cause very rapid gyrations. But there must be at least some gyratory velocity in case of friction, since there must be some such velocity before friction comes into play.

Since as much air in general must flow from the center above as toward it below and the tendency of each is to cause gyrations the contrary way, the only force which there is to overcome the resistance between the atmosphere and the earth's surface depends upon the differences of the gyratory velocities above and below, as in the case of the general motions of the atmosphere, and the expression of this force is similar to that of (44) in the preceding chapter. This, in the case of an ordinary cyclone in which v in the interior is greater below than above, is positive in the interior part where the air rises, and the contrary in the exterior part where it descends to the surface. Hence in the northern hemisphere the gyrations in case of friction, as in that of no friction, are positive in the interior and negative in the exterior part of the cyclone, just as in the case of the general motions of the atmosphere, they are eastward in the higher latitudes and the contrary in the lower latitudes, as explained in § 155. But the relations between the areas occupied by the two kinds of gyrations and the relative velocities are determined in this case upon a different principle from that by which they are determined in the case of no friction in § 189. They must in this case satisfy the following condition, similar to that of (45), § 156,

$$(17) \qquad \int \rho F_v d\sigma = \int \rho^2 F_v d\rho = 0$$

the integral of which must extend over the whole area of the cyclone, or in the last form from the center to the circumference. This condition makes the sum of all the moments of gyration arising from the action of the air through friction upon the earth's surface equal 0, as they must be, since the absolute forces arising from the temperature gradient are in the directions of the radii, and hence cannot give rise to such moments, the deflecting forces being of the nature of centrifugal forces, and consequently not real forces, § 143. On account of the complexity of the problem and the uncertainty in the law and amount of friction, we cannot obtain any quantitative results, but can merely infer that where there are gyrations in the one direction in the interior, there must be counter gyrations in the exterior part. A cyclone is usually understood to be the interior part, consisting of very rapid gyrations, manifesting themselves by very strong and often destructive winds. The exterior part, therefore, consisting of comparatively very gentle gyrations, usually not distinguishable on land from the various abnormal and more local disturbances, is properly called *the anticyclone*, both together forming one system. Where the former exists we know the latter must, although it may not be observable.

Central calm and calm zones.

194. There are three portions of a cyclone where there is little force to produce and to sustain a motion of gyration at the earth's surface. According to (44) of the last chapter, which is applicable to cyclones as well as to the general motions of the atmosphere, the force which overcomes the friction of the gyrations at the earth's surface depends upon dv, and this cannot have a value unless there is an ascending or descending current and v has a value which either increases or decreases with increase of altitude. But by (14) $v'=v=0$, at the center, where by (6) $\varphi=0$, and at the outer limit of the cyclone where $\varphi=180^\circ$. Hence very near both the center and the outer limit of the cyclone the force F_v must be too weak to overcome the friction of any sensible gyratory velocity. Hence there is always a calm at the center of a large cyclone, which is often observed to be 15 or 20 miles in diameter.

There is also an intermediate zone between the center and outer limit, where, for the same reason, there cannot be a gyration at the earth's surface. For since the air, in an ordinary cyclone, ascends in the interior and descends in the exterior part, there must be an intermediate zone where there is no sensible ascent or descent and where, consequently, there cannot be any sensible gyratory velocity, since, by (44), $F_{v'}$, the force which overcomes the frictional resistance to such motion, sensibly vanishes there. This is in accordance with what has already been deduced, for if there are gyrations in the interior part in the one direction and on the exterior part in the contrary direction, then there must be an intermediate zone where there are no sensible gyrations.

Relation between the motions of upper and lower strata of the atmosphere.

195. In an ordinary cyclone the radial motion of the air in the lower strata is *toward* the center, and in the upper ones out *from* the center, while at the same time the air in all the strata gyrates, either cyclonically or anticyclonically, around the center. The normal components of velocity v, especially a little away from the earth's surface, have the relation very nearly of (14), by which the positive values of v at the earth's surface, denoted by v', are decreased, and the small negative values in the exterior part increased, in proportion to increase of altitude h. The velocities and directions, therefore, in the lower strata a little above the earth's surface must be somewhat as represented by the heavy arrows, and in the upper as represented by the dotted arrows, in Fig. 2. By (14) the small positive values of v in the interior near the surface are gradually decreased with increase of altitude and become negative before the stratum is reached at which the inward flow toward the center is changed to an outward one, and consequently before the directions

incline from the center, but everywhere above that stratum the inclination is outward, and the greater the altitude the greater the velocities represented by the dotted arrows.

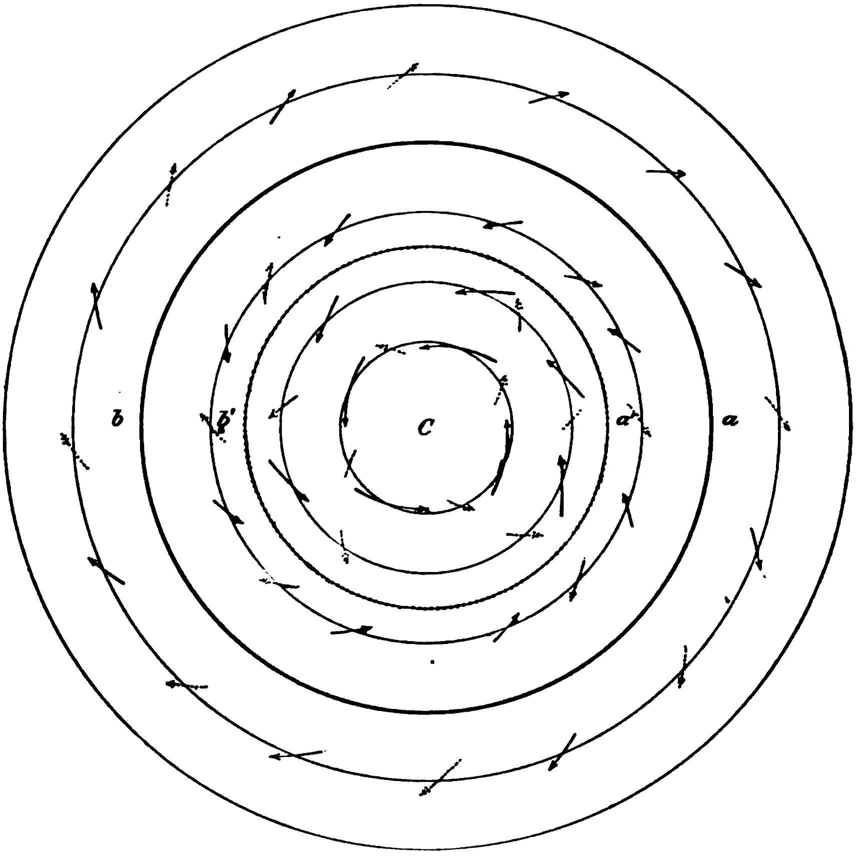


FIG. 2.

At and near the earth's surface the friction is so great that in general the deflecting forces are not sufficient to cause very great velocities, except very near the center; but these forces, arising from the flow of the same amount of air outward from the center above where there is comparatively little friction, would soon cause a very great gyrating velocity the contrary way if it were not for the limit imposed by (14), for as soon as that limit is reached there is no force to keep up the interchanging motion between the interior and exterior parts of the cyclone. The whole system acts as a governor in machinery. If the interchanging motion is a little too great, the gyratory motions and the forces arising from them are so increased at once as to diminish this interchanging motion, and if this motion is a little too small the reverse takes place.

The altitude at which the gyratory or normal component of velocity v changes sign and becomes negative depends upon the value of v' at the earth's surface and upon the energy of the cyclone or value of A_1 in (14). Hence it may be only at a very high altitude that the gyrations are reversed and become anticyclonic in the interior of the cyclone, as represented in Fig. 2. If this occurs before the plane is reached which divides the lower strata of inflowing air from those of the outflowing air above, then the anticyclonic gyrations have an inclination of the winds toward the center, and this only becomes outward in the strata of outflowing air as represented in the figure.

The preceding general deductions from theory are in general verified by observation. According to the conclusions at which Dr. Hildebrantsson arrived with regard to the upper currents of the atmosphere, from the synoptic chart of the Royal Meteorological Institute of Denmark, very near the center of a depression or barometric minimum the upper currents move very nearly parallel with the isobars and lower currents, but with increase of distance they incline from the center and the directions are to the right of those of the currents below. This is at altitudes where they have not become anticyclonic.

Mr. Ley, from 620 observations upon the motion of cirrus clouds, arrived at the following general law showing the relation between the direction of the higher currents of the atmosphere and the distribution of atmospheric pressure at the earth's surface: "*The higher currents of the atmosphere, while moving commonly with the highest pressures, in a general way, on the right of their course, yet manifest a distinct centrifugal tendency over the areas of low pressure and a centripetal over those of high.*"[18]

These observations having been made on the directions of the motions of cirrus clouds, which are always at very high altitudes, the "centrifugal tendency" arises from the radial outward motion in these very high strata, the altitudes of which, however, and the distances from the center were not such as to cause the anticyclonic motions represented in Fig. 2.

Mr. Ley further states, that "there occur at rare intervals in Western Europe depression systems, which affect, but in a very singular way, the directions of the upper currents, reversing them so that they become, on all sides of the area, nearly, or quite, in opposition to Ballot's law; that is to say, there exists a direct (cyclonic) upper current circulation above a retrograde (anticyclonic) circulation of the surface winds."

These cases, at rare intervals, are the extreme cases in which the altitude is so great that the radial motion of the cirrus clouds is not only outward, but also the cyclonic gyrations are reversed. The motions in this case are represented by the dotted arrows in the middle part of Fig. 2, in which it is seen that the motions are nearly in a direction contrary to that of the currents below.

From an examination of 121 cases of high winds on Mount Washington, Professor Loomis came to the conclusion (1) that high winds there circulate about a low center, as they do near the level of the sea, and (2) that the motion is nearly at right angles to the direction of low center. These being high winds must have been mostly near the center, and the motions are in accordance with the cyclonic theory.

Inclinations at and near the earth's surface.

196. The inclinations of the gyratory motion in a cyclone at some distance above the earth's surface are determined by the interchanging motions, and are almost entirely independent of friction, which is here very small and depends upon differences of velocities rather than upon absolute velocities, as at and near the earth's surface. Whatever the law and amount of friction in the strata generally above the earth's surface, the air in an ordinary cyclone must necessarily move toward the center below and from it above, and this determines there the direction of the resultant of the gyratory and radial motions. At and near the earth's surface, however, where friction is a function of the velocity which is somewhat proportional to the velocity, these directions are governed more by the law and amount of friction.

If in (48) and (49) of the preceding chapter we put $\cos\theta=1$ and $n\cos\psi$ for n, they are applicable to cyclones. The latter of these expressions may be supposed to hold approximately at and near the earth's surface, since friction there is so great in comparison with the term arising from the mere inertia of the air that the latter may be neglected in comparison. With the changes indicated above this becomes, in the case of cyclones,

$$\tan i=\frac{f}{2n\cos\psi+\nu} \tag{18}$$

in which ν is determined by (3).

From this expression it is seen that i depends upon friction, and that the greater this is the greater is the value of i. At the surface, therefore, where friction is comparatively great, the direction of the resultant motion is very much more inclined toward the center than it is a little above the earth's surface, and the value of i is greatest at the surface and diminishes with the altitude. The gyratory motion, therefore, is most nearly radial at the surface and gradually approximates to a circular gyration with increase of altitude.

The friction constant f remaining the same, it is seen from (18) that i must become small near the center, where the gyratory velocity is great, since in this case the value of ν by (3) becomes very great. Toward the center, therefore, at the distance where the gyratory velocity is great, the value of i becomes small and the gyratory motion nearly circular. At a distance where the air first begins to be drawn in toward the center, where ρ is large and consequently ν is small, the value of i is large and the motion is nearly radial, but the nearer the center the greater is the value of ν and the smaller the value of i, and the more nearly circular is the gyratory motion.

Since $n \cos \psi$ becomes small near, and vanishes at, the equator, so the value of i, all other circumstances remaining the same, increases with the decrease of latitude, and at the equator, where $\cos \psi = 0$ and ν also vanishes, it becomes either 90° or 270°, according to the sign of u or direction of the radial motion at the surface, and the motion of the air then is entirely radial and there can be no gyrations.

As friction is less on sea than on land (18) gives a smaller value of i, and consequently more nearly circular gyrations, on the former than on the latter.

The average inclination in large cyclones, obtained by Mr. Ley from a great number of observations made at fifteen stations, mostly of the British Isles, is 20° 31′. He also found the inclination greater at inland than at well exposed coast stations. The collective mean for Brest, Scilly, Farmouth, Pembroke, and Hollyhead is 12° 49′, while for the inland stations, London, Nottingham, Oxford, Brussels, and Paris it is 28° 53′.[13] This indicates that the inclination is greatest at the inland stations, where the friction is the greatest, for at the coast stations the gyrations were in a great measure over a water surface.

Subsequently Mr. Ley determined the relation between the directions of both the upper and under currents of the atmosphere and the radius drawn from the center in areas of low barometric pressure from the averages of a large number of observations in the British Isles, France, Spain, Switzerland, Austria, Turkey, Russia, Denmark, Sweden, and Norway.[14] The average inclination given by all for the under currents is 24° 7′, which is a little greater than his other average, as it should be, since there was a greater proportion of inland stations, and the friction, consequently, on the average, greater.

From the Weather Maps of the United States Signal Service, for the years 1872 and 1873, Loomis obtained[15] an average inclination of nearly 47°, which is much greater than either of the averages obtained by Mr. Ley. The average latitude being less, the inclination from what has been shown should be some less, but the difference is too great to be entirely accounted for in this way.

The remaining part of the difference may be explained in another way. Mr. Ley's observations were mostly of winds of considerable force, but Professor Loomis took in all within the isobar of 29.9 inches, which included many observations of velocities of only 3 or 4 miles per hour. The average distance from the center, therefore, of the observations used by Professor Loomis was, no doubt, much greater than that in the observations used by Mr. Ley. The average inclination, therefore, obtained by the former should be the greater, for reasons which have just been given.

From a subsequent and more thorough sifting of numerous cases taken from the Weather Maps of the Signal Service Loomis[25] deduced an average inclination for low barometer a few degrees less than that given above. From this discussion it resulted that inclinations at small distances from the center are less than for greater distances, as, we have seen, theory requires. The inclinations for low barometer ranged from about 37° at a distance of 250^{km} to 44° at $1,200^{km}$.

The average inclination obtained at the same time for low barometer from Hoffmeyer's charts of the northern part of the North Atlantic is considerably less than in the preceding case, as it should be, being a water surface, and the friction less; and the inclinations likewise increased with increase of distance from the center, being 25°.4 at the distance of 278^{km}, and increasing to 34°.8 at the distance of $1,535^{km}$. For barometric pressures above 760^{mm} the inclinations were still larger.

The average inclination obtained by Captain Toynbee from the discussion of the observations in the North Atlantic during the great cyclone on the 24th and the 25th of August, 1873, is 29°, the observations having been made on the average at the parallel of about 50°. This result, although on the ocean, is greater than those obtained by Mr. Ley for the land, though much less than that of Loomis.

The observations of Padre Viñes[13] on the hurricanes of the Antilles confirm the preceding deductions from (18). In all these hurricanes it was observed that the gyrating winds cease to be circular at a long distance from the vortex, and are found to deviate from the tangent to the circle with an inclination toward the center, forming a kind of spiral. This converging is likewise said to "vary, not only in different hurricanes, but likewise in the same hurricane, with different directions and intensities of the wind and with different distances from the vortex." In the same connection it is stated that it is "especially small at no great distance from the vortex."

In more recent researches Hildebrandsson has deduced from observations at Upsala an average inclination of 40°.2 and for the average of the three maritime stations, Wäderobod, Utklippan, and Sandön, 32°.2. The inclination at Upsala at all distances from the center was found to be greater in the summer than in the winter.[16]

Since the gyratory velocity *v* gives rise to the force which drives the air from the central part of a cyclone, and hinders its flowing back toward the center with a velocity due to the gradient of pressure, if the gyratory motion is over an uneven surface, and especially if there are mountain ranges and valleys in the direction of the radius of the cyclone, this gyratory velocity in the valleys, and even on the lee-side of a single mountain range, cannot exist, and hence, in such cases, there is a very strong current in the valley or on the lee-side of the range, due to the difference of pressure at the external and interior parts of the cyclone, caused by the gyrations above the mountain ranges. The velocity of the current in this case is that due to the full effect of the barometric gradient where there is no hindering force arising from a motion or component of motion transverse to the direction of the radius. In such a case the direction of motion and the inclination are determined by the direction of the mountain ranges, and the motion may be directly toward the center or vary considerably to either side of this direction.

Effects of gyration on pressure.

197. From (50) of the preceding chapter, by putting $\cos\theta=1$ and $n\cos\psi$ for n, for reasons heretofore explained, and neglecting the last term depending upon friction and the inertia of the air, we get the following expression applicable to a cyclone:

$$(19)\qquad \frac{d\log P}{\alpha dU}=\left(\frac{2n\cos\psi+\nu}{\cos^2 i}\right)\frac{v}{}$$

According to this expression the pressure at any altitude is a maximum where $v=0$. But by (14) this depends upon v', and the greater its value in the interior of the cyclone the greater the altitude of the stratum where $v=0$, at any given distance from the center. In Fig. 3, let the ordinates of the unbroken curved line, as indicated by the arrows perpendicular to the radius of the circle below, intended to represent the base of the cyclonic system, represent the values of v', and those of the dotted curves, of which only two are given, the values of v for strata above the earth's surface, the curve with the greater negative values of v corresponding to the higher stratum. Then the strata of equal pressure are represented by the dotted curved lines in the upper part of the figure. These, on account of the higher temperature and less density of the air, are farther apart in the central than the external part of the system, so that at a certain altitude, however concave they may be at the earth's surface in the central part, they are convex even at the center, and the higher the more so, just as they are at the equator in the general motions of the atmosphere, as represented by Fig. 6, § 159.

At the surface the greatest pressure is at *e*, the distance from the center at which $v'=0$, and at which there is no gyratory velocity at the earth's surface. For a stratum above the earth's surface of which the values of v are represented by the ordinates of the first dotted curve below, and of which $v=0$ at the distance *c f* from the center, the maximum pressure is at the same distance. And so the higher the stratum the nearer to the center is this maximum, until above a certain height the maximum is at the center. Above this altitude the values of v are negative at all distances from the center, as represented by the second of the dotted curved lines below, which does not intersect the line of abscissas except at the center. The maximum of the pressure, however, does not in general coincide with the maximum vertical ordinate of the dotted curve in the figure, representing the stratum of equal pressure, though this is very nearly the case near the earth's surface.

The altitudes at which $v=0$, and the greatest pressures in any horizontal stratum occur, is a line commencing at *e* and inclining toward the center, somewhat as represented in Fig. 3. These lines, as seen by 14, depend upon the values of v' at the earth's surface, and are consequently very different in different cases, but they must at some altitude come together at the center. For all parts within and under these lines in a vertical section of the cyclonic system the gyrations are cyclonic, but for the parts without and above they are anticyclonic, that is, the contrary way, just as in the case of the general motions of the atmosphere for all parts under the curved line *e c*, Fig. 6, § 159, the gyrations are from east to west, but without and above from west to east.

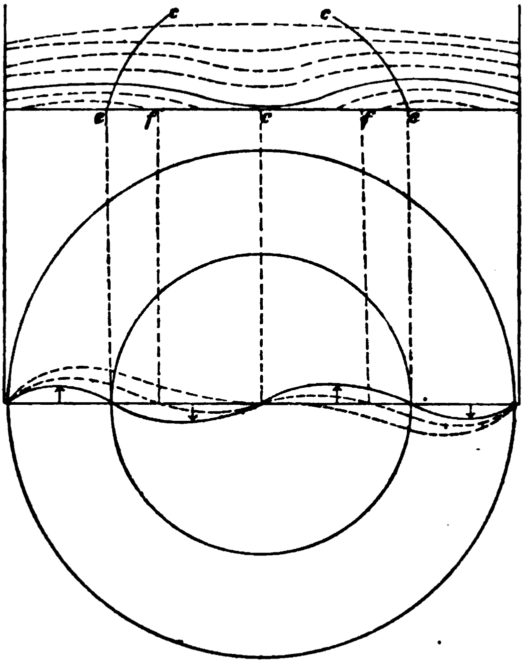


FIG. 5.

As the air flows into the interior of a cyclone below, its tendency to run into a gyration around the center is very much counteracted by the friction of the earth's surface on the one side and that of the strata above, which have a gyratory motion, at least relatively, the contrary way. But when this air rises in the interior and central part up to the upper regions, where there is no friction from the surface of a solid such as that of the earth's surface on the one side, and comparatively little between the strata of different gyratory velocities on the other, the tendency to run into a gyration the contrary way in flowing out *from* the central part is very little obstructed until it comes to have an anticyclonic gyration determined by (14); for if it should have a greater one the deflecting force arising from it, according to the principle of §145, would entirely prevent the outward flow.

It is seen from (19) that the increase of pressure from the center outward depends upon the deflecting force of the earth's rotation, the gyratory velocity of which is n, upon the polar distance ψ of the center of the cyclone, and upon the centrifugal force arising from the gyrations, which is $\nu v = v^2 : \rho$. In the interior of the cyclone both these forces are in the same direction, and outward from the center, and hence they cause a very steep gradient of increasing pressure, especially near the center, where ρ is small and consequently the centrifugal force very great. But in the anticyclonic part of the system these forces are comparatively small on account of the smallness of the gyratory velocities and the centrifugal force, and also on account of the great distance from the center. Moreover, these forces here are in contrary directions. The motion of gyration being here reversed the deflecting force due to the earth's rotation is also reversed, and consequently is toward instead of from the center, while the centrifugal force here though small, is still outward. There is, therefore, in the anticyclone, only a very small gradient of pressure, decreasing toward the outer limit.

The diminution of pressure in the interior is being constantly observed in the passage of all storms of a cyclonic character, and the amount, of course, is greater or less according to the violence of the gyratory motions produced by the conditions, but a depression of one inch or more of the barometer at sea-level below the normal pressure is often observed. In the great storm which passed over Scotland January 26, 1884, there was a reading at Ochtertyre, near Crieff, of 27.222 inches when reduced to sea-level. This is supposed to be the lowest reading ever made in Europe. In a cyclone of so great violence the readings in the zone of high pressure would, no doubt, have been much above those of a normal pressure, and the difference between the barometric pressures of this zone and the center must have been at least 3 inches.

In the exterior of the system, or anticyclone, the gradient, for reasons just given, is small, and on land, as in the case of the corresponding anticyclonic gyrations, is generally not observable on account of the numerous abnormal disturbances producing greater local gradients. Before all great storms, however, it is frequently observed that the barometer stands unusually high, and the same soon after it has passed. This was first noticed and remarked upon by Redfield. These are at the times when the ring of high pressure around the cyclone passes over the plane of observation. At Havana, in Cuba, over which the tropical cyclones of great violence frequently pass, the zone of high pressure is distinctly observed, both before and after the passage of the central part of low pressure and great violence, and the preceding abnormally high pressure is usually regarded as an indication of the approach of a cyclone. The approach of the hurricane of September, 1875, was indicated at Havana by a sudden rise of the barometer while the cyclone was yet at the Windward Islands, about 1,200 miles distant. Also, on the 13th of September, 1876, there was a great rise of the barometer at Havana while a cyclone was causing great destruction on the island of Porto Rico.[12] The region from which these cyclones come being mostly over the ocean, with few disturbances from land surface with hills and mountains, the conditions are more regular than on land and the friction less. Consequently these cyclones are better developed, and the gyratory velocities and resulting gradients are generally very great, notwithstanding the low latitude.

It is seen from (19) that the lower the latitude and the smaller the value of cos ψ the less the gradients, and, for the same extent, the smaller the amount of depression in the center of a cyclone with the same gyratory velocity. As the whole depression depends upon the integration of the gradient of decreasing pressure from the zone of high pressure to the center, of course the amount of depression depends upon the extent of the cyclone as well as upon the rapidity of the gyrations and the latitude. The gradients, therefore, everything else remaining the same, must be the greatest at the poles, gradually decrease with decrease of latitude, and vanish at the equator. Hence at the latter there are scarcely any perceptible abnormal irregularities, and only the normal diurnal changes, occurring regularly every day, are observed. Even in the center of areas of great rainfall, which are usually accompanied by barometric depressions at other places, there is no sensible diminution of the pressure.[11]

On account of the large amount of friction at or near the earth's surface, the gyratory velocities, and the consequent lateral pressures on each side of the zone of high pressure, which cause the heaping up of the air, would be very much diminished there, and likewise the lateral pressure on both sides toward this zone, if it were not that as soon as this pressure is diminished the air begins to flow out from beneath this zone of high pressure, on the one side toward the center and on the other toward the outer limit of the system, the deflective forces of which depending upon the earth's rotation, § 145, tend to keep up and strengthen these gyrations on both sides. In fact, without this outward flow near the earth's surface there could be no anticyclonic gyrations very near the earth's surface in the exterior part, as represented in Fig. 2 and Fig. 3, and an outward inclination of the resultant directions of motion. At only a moderate elevation above this comparatively thin stratum of outward flow, where the very gradual motion of interchange is still toward the center, the inclinations are slightly inward, until you ascend to the stratum which divides the strata in which the air flows toward the center from those in which it flows from the center.

With the regular assumed conditions of a cyclone, the isobars and whole area of depression would be circular; but on account of the greater easterly velocity of the upper currents of the atmosphere, over the United States, and in middle latitudes generally, they are elongated in the direction of the general motions of the atmosphere in which the cyclone is. Loomis[19] found the average ratio of the longer and shorter axes to be 1.94, and the average direction of the greater axis N. 39 E. in the United States, and the ratio 1.60, and the direction N. 31 E. in Europe.

Cyclones with a cold center.

198. If for any reason the central part of the atmosphere covering any circular area of the earth's surface is maintained in any way at a lower temperature than that of the surrounding part, and the temperature increases symmetrically on all sides from the center outward, so that it can be expressed by (5) with a negative value of A, it is readily seen from (15) that the interchanging motion in this case between the central and external parts of the area of disturbed temperature is reversed, and is from the center below and toward it above. According to (44) §155, however, which is applicable in case of cyclones, the gyrations at the earth's surface must be in the same direction as in the case of ordinary cyclones, since these gyrations depend upon Fv', which has the same sign in both cases. In the case of an ordinary cyclone, v decreases with increase of altitude, and as the vertical current in the interior in this case is an ascending one dv there is negative, and consequently Fv positive. In the case of a cyclone with a cold center, the sign of A, in (14) being changed, v is greater than v', and the gyratory velocity increases with increase of elevation, so that, since the vertical current is a descending instead of an ascending one, we still have Fv' positive in the interior gyrations and negative in the exterior ones.

Since v in this case increases with increase of altitude the distance, ρ, from the center at which v vanishes and changes sign, and at which the pressure is the greatest, increases as the altitude is increased, so that the curved line $e\,c$, Fig. 3, which represents the altitude at any given distance at which $v=0$, and the pressure is a maximum, inclines outward from the place of greatest pressure at the earth's surface. In this case, and for all altitudes at which v has no positive value at any distance from the center, the maximum is at the outer limit of the system. At all places within and above such lines in any vertical radial section the gyrations are cyclonic; but at those without and under, they are anticyclonic. In the case of the general motions of the atmosphere for either hemisphere we have the best example of such a cyclone, in which the pole of the earth is the center and in which there is a gradient of regularly increasing temperature from the pole to the equator. It is true there are some differences arising from the difference in the extent of the area considered, and the convexity of the surface which has to be taken into account in the case of the whole hemisphere, but the general results are the same. Fig. 6, therefore, of the preceding chapter, is approximately applicable in the case of a cyclone with a cold center, if we suppose the lines representing a section of the strata of equal pressure to be horizontal and the distance from P to E to represent the radius of the circular area covered by the system. In this figure the curved line $e\,c$ inclines toward the outer limit, as, from what is shown above, it must in a cyclone with a cold center. This line in either case does not necessarily have a maximum elevation before coming to the limit, since this depends very much upon the value v' at the earth's surface; but this is especially the case in a cyclone of this sort of moderate dimensions, since in this the deflecting force of the earth's rotation does not vanish at the outer limit, as it does at the equator in the hemispherical cyclone.

At all altitudes above the curved line $e\,c$, and all distances from the center between f and e, Fig. 3, the gyrations are anticyclonic, and the area or annulus of high pressure, as measured at the surface, is to the left instead of the right of the direction of motion in the northern hemisphere, contrary to Ballot's law; but if the pressure is taken at the altitude of the observed current, it is still in accordance with the law, since the maximum pressure in a horizontal stratum of any altitude is in the line $e\,c$.

In the case of a cyclone with a cold center, the outflow of air near the earth's surface from beneath the zone of high pressure toward the center is contrary to that of the interchanging motion, which is there from the center, and the resultant motion is composed simply of the excess of the former over the latter, combined with the gyratory motion, and gives a direction of motion inclined toward the center. But this, as in the case of the southwest winds in the middle latitudes in the general motions of the atmosphere, extends up only to a very moderate altitude, above which the inclinations are outward from the center, until arriving at the middle plane which divides the interchanging motions flowing in contrary directions, after which the direction of motion is slightly inclined toward the center. In the external part the outflow from beneath the zone of high pressure combines with the current of the interchanging motion of the earth's surface, which is outward just as it does in the case of the general motions of the atmosphere, by which the trade winds are strengthened and the outward inclination of resultant directions increased.

In an ordinary cyclone the force by which the air at the earth's surface where friction is great is drawn in toward the center is much greater than in the case of a cyclone with a cold center, and hence it is drawn in near to the center before it ascends in the interior, and the increasing moments of gyration causes very rapid gyrations near the center where the radius of gyrations becomes small. But in one with a cold center this force is comparatively weak and the air rises up into the upper strata mostly at considerable distances from the center, so that there is little concentration of energy near the center, as there is in ordinary cyclones. The sum of the positive and negative moments of gyrations, however, taken separately in (17) are the same in both cases where the velocity of interchanging motion is the same, but they are differently distributed, in the case of an ordinary cyclone those near the center where the gyrations are rapid, being small on account of the smallness of ρ.

Progressive motion of cyclones.

199. The directions in which the centers of different cyclones move varies very much, but with very few exceptions they are always easterly. The average direction for the United States, as determined by Loomis from an examination of the individual directions laid down on the weather maps of the Signal Service for three years, comprising four hundred and eighty-five cases, is N. 81° E., with an average velocity of 26 miles per hour. These averages for each month indicate a small annual inequality. For April to September, inclusive, it is N. 82.5° E., 23 miles, and for October to November, inclusive, N. 78.5° E., 29 miles per hour. Hence the direction is more easterly and the velocity greater in winter than in summer. There was also found to be a diurnal inequality in the velocity of progress, this velocity being about one-fourth greater on the average from 4.35 p. m. to 11 p. m. than it is for the remainder of the day.

The principal cause of the progressive motions of cyclones is undoubtedly the general motion of the atmosphere in which the cyclone exists. The velocity of progressive motion, however, is generally much greater than that of the general motion of the atmosphere, even up at a considerable altitude. From the table of § 161 we get for the middle latitude of the United States a velocity of 26 miles per hour at the altitude of about 2.5 miles. If, therefore, the easterly progressive velocity is the sole cause of the progressive motion in the United States, the center of the controlling power of the cyclone must be at that altitude. The principal part, however, of the rate of progress of a cyclone must be due to that of the general motion of the winds, and the annual inequality in the former to a similar one in the latter. According to § 169 the easterly velocity of the general motions of the atmosphere is about twice as great in January as in July, and, hence, taking the average velocities for the colder half and the warmer half of the year separately we should have very nearly the ratio of 29 to 23, which is that given above between the average rate of the progressive motions of cyclones for the corresponding halves of the year.

The directions and velocities of progressive motion are also much influenced by the irregularities and deflections of the general motions of the atmosphere, caused by continents and mountain ranges. From this cause, we have seen, § 171, the westerly motion in the lower latitudes is deflected around toward the pole on the east side of each of the continents in both hemispheres. From this cause there is a polar component of velocity over the whole of the United States east of the Rocky Mountains, which extends up into high latitudes. This, combined with the eastward component, gives a progressive motion of cyclones with a direction a little to the north of east, in accordance with observation. Since the velocity of the general motion of the atmosphere in which the cyclone exists is generally greater in the upper than the lower strata, at least in middle and higher latitudes, the progressive motion of the cyclone must depend very much upon the altitude of the center of power of the cyclone. If the air is entirely or nearly saturated with aqueous vapor this center must be down nearer the earth's surface, and the progressive velocity of the cyclone depend more upon the general velocity of the air nearer the surface than in the case in which the vapor has to ascend to a considerable altitude before condensation and the giving out of latent heat commences. In the latter case the center of the power controlling the progressive motion of the cyclone is more in the upper strata, and the velocity of progression should generally be greater.

In the progressive motion of a cyclone it must not be supposed that there is an actual transfer of the air which is gyrating at any one time, but rather that there is a continual forming of a new cyclone a little in advance, in which the air in the advanced position, especially of the lower strata, is drawn into a gyratory motion by the power of the cyclone in higher strata where the vapor is mostly condensed, while the gyrations of the part left in the rear are gradually destroyed by friction. The air above, rarefied by the latent heat of condensation, is carried along much faster generally than that near the earth's surface, and, consequently, leaves the gyrating air near the earth's surface behind, while other air is brought into action under the center of power above in its advanced position.

As the power of the cyclone is mostly in the aqueous vapor condensed, and without this we rarely have the conditions of more than an initial cyclonic action, the velocity and direction of the progressive motion of a cyclone depends, to some extent at least, upon the distribution of this vapor in the region in which the cyclone exists, and the cyclone is likely to be drawn somewhat in the direction in which there is the most vapor. For this reason it is, perhaps, that the chain of lakes between Canada and the United States seems to be a great highway for cyclones.

In the middle latitudes, where cyclones have an easterly tendency, the warmer and moister air is on the southern side. This, in the lower strata, is drawn around in the gyrations to the eastern side, while it at the same time ascends to higher altitudes where the moisture is condensed. If it should be carried around to the northeast side before it rises to the altitude where it is mostly condensed, the effect would be to change the direction of the progressive motion, due mostly to other causes, a little toward the northeast, but, on the other hand, if it should be carried around to the southeast only before it has ascended to the altitude of principal condensation, then the effect would be to change the direction a little to the southeast. The vapor carried around in this way from the south to the east side tends continually to give rise to a new cyclone a little in advance of the old one, and so to increase the velocity as well as change the direction of progressive motion.

It is seen from what precedes that both the velocities and directions of the progressive motion of the cyclone may depend upon several circumstances besides the motion of the atmosphere in which it is, and hence there are great observed differences in both. In cases, however, in which there is an apparent westerly motion, the cause is most probably the formation of a new cyclone a little in the rear and the dying out of the old one, so that this retrograde motion is only apparent and not real.

Veering of the wind and changes of pressure.

200. If the atmosphere of any region did not have a progressive motion there would be no veering of the wind, and the barometric depression would increase until the cyclonic action had arrived at its maximum, after which there would be a gradual decrease until it had died out; but where the whole system, in its progressive motion, passes over a place, the occurring phenomena are different. Let the arrows of Fig. 4 represent the directions and relative velocities at the earth's surface in a cyclone of the middle and higher latitudes, the interior full circle representing the annulus of highest barometer. For places near the equator the inclinations from the isobars in towards the center would have to be considerably greater. If the cyclone passes centrally over the place of observation, say from west to east, while this place is in the outer or anticyclonic part, as at *a*, there is a very slight rise of the barometer and a very gentle wind from a point a little west of north, but not generally observable for reasons which have already been given, § 193, except occasionally in the comparatively regular and well-developed cyclones of the West India Islands. Then comes the highest barometric pressure and the calm belonging to it, § 194, and when the place of observation is at the point *b* in the cyclone the cyclonic winds are observed with a direction toward and around the

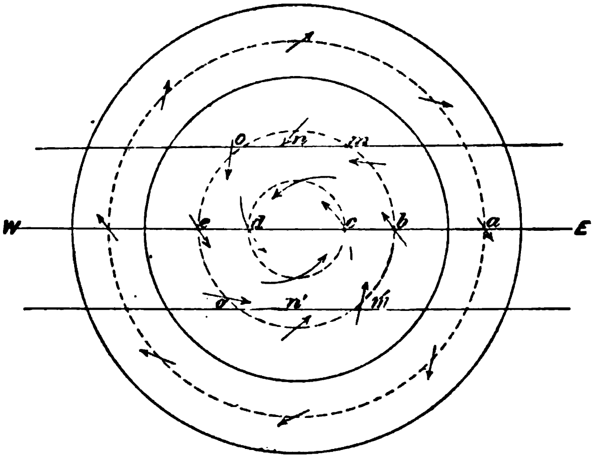


FIG. 4.

center, the velocity of which continues to increase with little change of direction until at some point (*c*) still nearer the center, where the gyratory motion is the most rapid. While the center and parts very near on either side are passing over the place there is a calm, and the barometric pressure is then the least. When the progressive motion continues a little longer, until the place of observation is at *d*, there has been a complete veering of the wind to the contrary direction and the barometer is now rising. If the direction of progressive motion is exactly from west to east this change is from about S.SE. to N.NW., but if, as usual in the middle latitudes of the northern hemisphere, the cyclone moves toward the E.NE., then the veering, it is readily seen from the figure, must be very nearly from SE. to NW. Within the tropics, where the direction of progressive motion is W.NW., the manner of veering is readily seen by placing the figure with the line *W. E.* in that direction, when it is seen from the directions of the arrows that the veering is from a north to a south wind, or very nearly. The same may be done for any other direction. When the place of observation in any case is at *e* the gyratory velocity and violence of the wind has very much diminished and the height of the barometer increased, and soon after follows the highest barometer and the calm in that part of the system. If the area of central calm is small and the gyrations are very rapid up close to the center, as in violent cyclones of moderate dimensions at sea, then there is a very sudden change of a very strong wind to the opposite direction.

If the north side of the cyclone passes eastward over any place of observation in the northern hemisphere, so that the place occupies successively the places within the cyclone *m*, *n*, and *o*, then it is seen that the change of the more violent winds in the cyclone proper is from about (at *m*) E., to NE. at *n*, and then N. at *o*. In this case there is a veering through only about one quadrant, and this is in a direction contrary to that of the hands of a watch; but if the south side passes in the same direction over the place of observation, so that it occupies successively the places *m′*, *n′*, and *o′* in the cyclone, then the veering is from S. at *m′* to SW. at *n′* and then to W. at *o′*, and the veering in this case is through one quadrant in the same direction as the motion of the hands of a watch.

The direction in which the wind veers depends entirely upon which side of the place of observation the center of the cyclone passes, and the amount of veering, upon the distance of the path of this center from the place of observation. If this distance is very small in comparison with the dimensions of the cyclone, the veering is through two quadrants nearly, but with greater distances the amount of veering is less, to the outer limit of the interior violent part of the cyclone, where it vanishes. For other directions of progressive motion, the initial and final directions in the veering of the wind is indicated in all cases by the arrows when the line *W. E.* is placed so as to coincide in direction with the direction of progressive motion.

The winds at any place are liable to veer as frequently on the average in one direction as the other, unless the cyclones pass more frequently on the one side than the other. In the middle latitudes of the United States and of Europe cyclones pass more frequently on the north side, so that in these latitudes the veering is most frequently in the same manner as the motion of the hands of a watch.

In the southern hemisphere the veering of the wind would be indicated in the same way in any case by means of a figure similar to Fig. 4, but with the arrows so inclined as to give a gyration in the contrary direction.

Gradual enlargement of cyclones.

201. In the preceding theoretical consideration of cyclones it was assumed they have definite and unvarying limits, but this is not the case in nature. The portion of air which is first brought into gyratory motion is continually acting by friction upon the contiguous portions, so that the interior gyrations encroach upon the exterior counter gyrations and the volume of ascending air is increased, and the outward flow of this above, upon which the counter anticyclonic gyrations depend, increase the volume and extent of the air in these external gyrations, so that there is a tendency to expansion, as long at least as the power which keeps up cyclonic action is not diminished.

It may be that cyclones mostly commence over a small area and gradually enlarge, but this is not necessarily so. If the initial temperature conditions of an ordinary cyclone, which consist of a portion of the atmosphere with a warmer interior and gradient of temperature decreasing outward, should extend over a large area, then the initial motions set up cover this same area, and the cyclone is commenced at once upon a large scale; but if the area of initial temperature conditions is of moderate extent only, and the cyclone has the power of continuance for a long time, it may gradually expand into a large cyclone, but it is doubtful whether this is the case where the initial conditions are confined to a very small area. The power of the cyclone dies out before the small cyclone can be expanded so as to cover a large area.

Tropical cyclones.

202. There is a class of peculiar cyclones which originate within the tropics, apparently near the polar limits of the equatorial calm belt, which at first are comparatively small, but they seem to have great power of continuance, so that they run through a long course and gradually expand as they go. Those of the North Atlantic originate almost entirely in the summer season, at the northern limit of the calm belt, far over toward the African coast. Being carried at first nearly westward by the westerly motion of the atmosphere in the trade-wind zone,

near the northern limit of the equatorial calm belt, until they arrive in the region of the West India Islands and Gulf of Mexico, they are there carried around over the United States or along the east coast or the route of the Gulf Stream by the atmospheric currents which are there deflected northward (§ 171) until they arrive in higher latitudes. Here their progressive motion is controlled by the strong easterly tendency of the atmosphere in these latitudes and other circumstances, as in the case of the cyclones which have their origin in these latitudes. Their progressive westerly velocity over the Atlantic is not nearly so great as that of the easterly motion in the higher latitudes, for although the westerly component of motion there at the earth's surface is considerable, yet we have seen by (59), §159, and the table of §161, that this westerly velocity gradually decreases with increase of altitude, so that at not a very great altitude it becomes not only small but reversed to an easterly velocity. The rate of progress here, therefore, is comparatively small, but it becomes still smaller while they are curving around into the higher latitudes, since the northerly component of velocity of the general air currents is small. When, however, they arrive into the middle and higher latitudes, where their general direction is a little north of east, they have the usual velocities of progression of cyclones generally in these latitudes. The path of the centers of these cyclones, therefore, is a kind of parabola having its vertex about the parallel of 30°.

According to Loomis,[19] the average direction and velocity of forty of these cyclones while moving westward was 24° north of west and 17.4 miles per hour. Of course the first part of their course is more westerly, and it gradually becomes more and more north of west, since there is a gradual curving around in these directions. In only two cases was the direction south of west. The average direction of these storms while traveling eastward to the parallel of 40° was east 38°.5 north, and the hourly velocity in this part of their course 20.8 miles. In the first part of this, however, the course was more northerly and gradually changed around to a course only a little north of east.

The steady current of the northeast trade-winds is not favorable to the origination of cyclones, and therefore they seem to originate at the northern limit of the calm belt only, or at least not so far within that the deflecting force depending upon the earth's rotation is too small to produce gyrations, for we have seen this force becomes very small near and vanishes at the equator; hence of the forty paths of cyclone centers examined by Loomis no part of any one of them was found within 10° of the equator.

At all seasons of the year except during the latter part of summer and the early part of fall, the trade-winds blow down to a latitude where the deflecting force of the earth's rotations is too weak to give rise to cyclones, and as they cannot originate within the zone of these winds, they can generally originate at the time only in which these winds do not extend down very near the equator. According to Laughton,[4] the following are the ratios in which these cyclones occur in the different months of the year:

January	1.5	July	7
February	2	August	28.5
March	3	September	24
April	2	October	20.5
May	1.5	November	5
June	3	December	2

In the South Atlantic the southeast trade-winds often extend considerably beyond the equator, and at any season too near to the equator for cyclones of this class to originate. Hence on the east side of South America and on the ocean adjacent, where there is a deflection of the general atmospheric currents by the continent and the range of the Andes around toward the South Pole, in a manner similar to that over the United States and the ocean adjacent, few, if any, cyclones of this class move around in a parabolic path into the middle latitudes of the Southern Hemisphere.

The east coast of China and the oceans adjacent, including Japan, is also traversed by numerous cyclones of this sort, originating in this same manner, and mostly at the same season of the year. These likewise curve around in a kind of parabolic path with its vertex near the parallel of 30°, until they arrive in the higher latitudes and assume a nearly easterly direction across the North Pacific. The average rate of progress is probably about the same as those of the North Atlantic and United States.

On the east coast of Africa and the adjacent ocean, including Madagascar, cyclones originating in the Indian Ocean near the southern limit of the equatorial calm belt when it has its most southern position, pass around into the middle latitudes of the Southern Hemisphere, where their directions become nearly easterly, being controlled there by the strong easterly general currents of the atmosphere. These cyclones, however, before they arrive at these latitudes, are remarkable for their small velocity of progression. This is because the velocity of the westerly current in the Indian Ocean, and of the currents deflected by the east coast of Africa down towards the lower latitudes, is small.

Rain and cloud areas in cyclones.

203. Since there are gently ascending currents in the interior of a cyclone with a warm center, this must be a region of cloud and rain, just as the equatorial calm belt is, where the air, coming in from both sides in the lower strata of the atmosphere, is deflected upward and becomes an ascending current. The explanation of the formation of cloud and rain is the same in both cases. Of course it is not to be supposed that the rain areas coincide in form and extent, except roughly and in a general

way, with the barometric depressions and areas of ascending currents, for the ascending and partially condensed vapor may be and generally is carried beyond by the stronger currents above into the anticyclonic area before it falls as rain; so that the center of the rain area is generally in advance of that of the area of low barometer, and the areas, as in the case of the isolars, are elliptical, and not circular, as they would be in case of regular conditions with no stronger general atmospheric currents above than below.

According to Loomis[20] "the form of these rain areas is sometimes quite irregular, but generally it approximates to an ellipse, of which the major axis is not quite double the minor axis." The average distance of the center of greatest rainfall from the center of low pressure, north of latitude 36°, was found to be 300 miles, but it sometimes exceeded 750 miles. The general direction of the longer axis of the rain area, and of the center of this area from that of the area of low pressure, is the same as the direction of the progressive motion of cyclones and of the longer axis of the area of low pressure. This indicates that they all depend mostly upon the general motions of the atmosphere, especially those of the strata at some distance above the earth's surface.

Considerable depressions were sometimes observed[20] where there was but little rainfall. This shows that the conditions of a cyclone may exist where there is not much moisture in the atmosphere. In such cases there is probably a large area with the interior part much warmer than the exterior, and with a large vertical gradient of temperature decreasing with increase of altitude, in which case the atmosphere approximates to the state of unstable equilibrium, which is an important condition of a cyclone. In such a case the ascending current in the interior would have to be continued perhaps a day or more before the interior part would become cooled down to the temperature of the exterior part, and the horizontal temperature gradient upon which cyclonic action depends destroyed.

In general, areas of low barometer are areas of cloud and rain, but there is no proportionate relation between the amount of rain and of barometric depression, since the latter depends very much upon the extent of the area, upon the deflecting force of the earth's rotation, and the amount of friction, § 197, while the former depends mostly upon the velocity of the ascending current and the amount of moisture in the air. Hence, with the same measures of temperature disturbance, and of consequent cyclonic action, there may be very different quantities of rainfall.

The rainfall also is often very unevenly distributed, for it rarely happens that the conditions in nature are such as to give a regularly formed cyclone. In fact, they are often only very rough approximations to such conditions. Besides, there may be a blending together of the effects of two or more of such sets of conditions, either contiguous, or the smaller ones contained within the largest. In the latter case we have smaller cyclones within a larger one, but all, perhaps, imperfectly developed and irregular. In such cases there are several centers of more violent rainfall within the general area of rainfall, and the isobars, also, are very much distorted and irregular.

While all areas of low barometer are areas, generally, of more or less rainfall, it must not be supposed that there may not be much rainfall without any sensible depression of the barometer. The atmosphere over a large area may be nearly saturated with aqueous vapor, and nearly or quite in a state of unstable equilibrium, and yet the temperature conditions over this area may not be such as to give an initial motion toward and around some central portion of considerable extent, but there may be slight, very local, conditions of this sort which give rise to ascending currents over small areas, for when the atmosphere is only very near the state of unstable equilibrium only a very slight determining cause in the temperature conditions is necessary to give rise to ascending currents and rapid condensation of vapor into rain, which may continue some time, though on account of the smallness of area there may not be a sensible fall of the barometer, even if there should be considerable cyclonic development, but even this latter is not probable in such cases.

When the atmosphere over a considerable area is in this state we have showery weather. As the whole region of atmosphere generally has a progressive motion, when the border of one of these areas arrives at any place a shower commences, which continues until it has passed by. If the central part of this area passes over the place the shower is of longer continuance, and the rate of rainfall may be very great while this part is passing, but if one side merely passes over it there may be a gentle shower only of short duration. If, while the atmosphere is in this state, temperature conditions from some cause are brought about, which determine the motions of the atmosphere below over a considerable area toward some one central part, this gives rise to a cyclone of considerable extent, which changes this state in a day or two and then fair weather follows.

Resultants of cyclonic and progressive motions.

204. As the part of the atmosphere in which a cyclone exists generally has a progressive motion, the combination of this with the gyratory motion gives a resultant motion which is different in direction and velocity on the different sides of the cyclone.

Let *a b*, Fig. 5, represent the velocity and direction of the cyclonic motion on the several sides of the cyclone, with its center progressing in the direction indicated by the arrow, and also let *b c* represent the direction and velocity of the general progressive motion, which may be assumed to be the same, or nearly, in all parts of the cyclone. It is seen from the figure that the resultants of the two motions have very different directions on the several sides. On the side of the cyclone on the right of the direction of progressive motion the velocity is changed from *a b* to *a c*, and is, therefore, much increased, though the direction is changed but little, since both motions are nearly in the same direction. On the left hand, on the contrary, the velocity is diminished, and the direction may be very different when the resultant velocity, *a c*, is very small. On the front side the direction is generally very much changed, since nearly the whole effect of the progressive motion is upon the direction of motion and not upon the velocity. The change may be so great here as to make the inclination negative or outward. In the rear the progressive motion changes both the direction and velocity, making the observed inclination of the wind there greater than on either of the other sides.

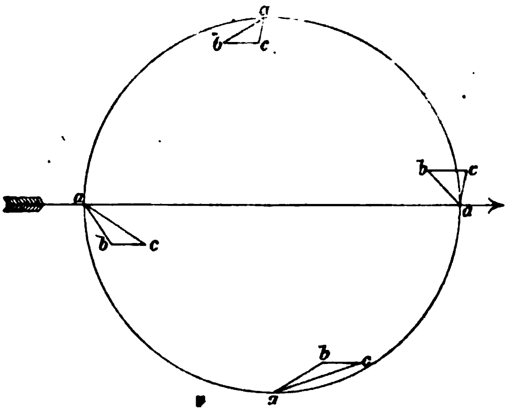


FIG. 5.

Since the velocity of the wind is increased the most on the right-hand side, this is called *the dangerous side* of the cyclone. But the greatest increase of velocity is in that part of the cyclone where the progressive motion coincides exactly with the cyclonic, which is somewhere between the rear and the right-hand side, according to the amount of inclination. In the cyclones within the tropics, where the inclination by theory and observation is great, the most dangerous part is pretty well back in the rear but on the right. In all cases the side exactly opposite the most dangerous part is the safest side, and the parts intermediate, differing one quadrant in direction from the center, the neutral parts, where the direction is changed mostly and the velocity but little.

Of course the relations between the resultant velocities and either of the components depends upon the relations between the latter. Upon land, where the progressive motion at the earth's surface may be very small in comparison with the gyratory, the resultant directions and velocities are not much changed, but this is generally different at sea, both in the trade-wind zone and in the middle latitudes of the North Atlantic, where the progressive motions are considerable. Fig. 5 perhaps more usually represents the relations found at sea, where the progressive motions may be nearly as great as the cyclone.

From an examination of the Signal Service Weather Maps Loomis[31] obtained results in accordance with theory. The inclinations on the average were found to be considerably less in the east than in the west quadrant, and the difference was still a little greater, as obtained by interpolation, between the ENE. and WSW. quadrants, which are somewhat the front and rear parts of cyclones in the United States. The differences are not so marked as represented in Fig. 5, and as they generally are at sea.

In the more regular and well-developed cyclones of the West India Islands the effect of progressive motion upon the resultants of velocity and direction is very observable in any one cyclone. The direction of the trade-winds here at the earth's surface is nearly westerly, and the velocity considerable in comparison with the velocities of cyclonic motion. The effect of this progressive motion, therefore, is to diminish the inclinations the most on the side of the cyclone, where the direction of cyclonic motion is from the north, which, since the inclination in these latitudes is considerable, is about at the northwest point of the cyclone and nearly in the middle of the anterior quadrant of the cyclone. Accordingly the direction of resultant motion here is observed to be nearly at right angles to the direction of the center, while at the opposite or rear part the direction inclines in very much toward the center.[12] But on the right-hand side the progressive motion is nearly in the direction of the cyclonic motion, and the resultant is nearly the sum of the two velocities. Here, therefore, the absolute velocity of the wind is the greatest, and this becomes the dangerous side of these cyclones, while on the opposite side the two motions somewhat counteract each other, and there is, consequently, comparatively little violence.[10]

If Fig. 5 is compared with Loomis's diagram of the resultants obtained for the directions of the wind for low barometer on Mount Washington,[22] it is seen that there is considerable similarity, but it would be almost perfect if the progressive velocities in Fig. 5 had been taken a little greater, and their directions, not from the west, but from N. 76° W., which was the average direction of the wind in the observations used by Loomis.

But although the observations of direction in the cyclones of the United States and the West India Islands confirm very satisfactorily the deductions from theory, yet those of Mr. Ley in England and France, for some unexplained reason, give contrary results. He obtained the greatest inclination for southeast winds and the least for northwest winds, which are nearly in the front and rear parts, respectively, of the cyclone.[14]

Also in Mr. Ley's subsequent and more extended researches, § 191, the average inclination in the front octant of the cyclone was found to be 37°, while that in the rear octant was only 12°.

Similar results were obtained by Hildebrandsson for the three maritime stations Wäteröbod, Utklippan, and Sandön. The greatest inclination, 28° 49′, was found on the east side, and the smallest, 12° 51′, on the west side.[24]

This variation from the results obtained by Loomis, and apparent contradiction of theory, is caused by the observed cyclones on the west coast of Europe being mostly, and especially in the winter season, on the east side of the large stationary cyclone having its center near Iceland. The part of the observed inclination due to the large cyclone, being far from the center, is larger than that of the smaller cyclone within it, and hence the inclination of the resultant is increased on the east side and diminished on the west side of the smaller cyclone.

For this reason the average inclination is greater in winter than in summer, as it has been found from observation.

In the upper part of the cyclone, instead of an inclination toward the center, there is a declination outward from the center. In this case, therefore, the effect of progressive motion upon the resultant directions and velocities is somewhat different, and the effect generally much greater on account of the comparatively great velocity of this motion.

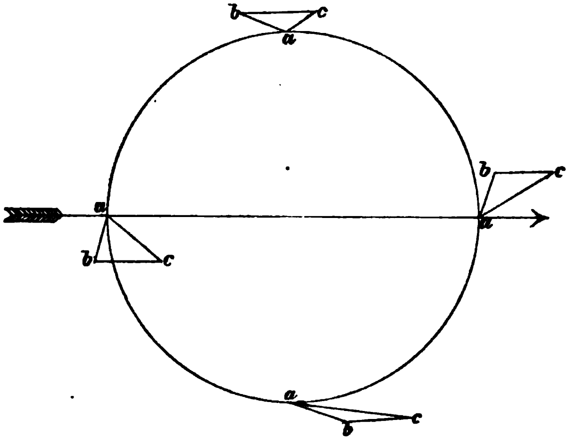


FIG. 6.

In Fig. 6 let *a b* on the several sides represent the cyclonic and *b c* the progressive velocities. The resultants, then, are represented by *a c*. It is seen that in the front and northeast quadrant the outward declination is very much increased and also the velocity, while in the rear the outward declination becomes an inclination toward the center with little increase of velocity. But the greatest increase of velocity is on the right-hand side and in the southeast quadrant, where there is little change of direction, while on the opposite side there is great change of direction with little change of velocity. On this side the progressive velocity and direction may be such as to almost or quite counteract and destroy the cyclonic motion, and hence this may often be a district of calms rather than of currents.

The resultant directions in Fig. 6 represent very nearly those obtained by Mr. Ley from observations of the directions of the upper currents of the atmosphere.[14] The average of more than three hundred observations in the front octant gave an outward declination of 53°, while in the rear one the average direction was nearly at right angles to the radius or progressive motion.

According to the estimates of Mr. Ley, from observations on the motions of the cirrus clouds, the most elevated ones not uncommonly traverse a distance of 120 miles in an hour, and occasionally much more. This great velocity occurs, no doubt, when the great velocity of the atmospheric currents in these regions (§ 165) coincides in direction with a great cyclonic motion, as in the south or southeast octant. On the other hand, he states that calms are uncommon in this elevated stratum, and that he observed only twice an actually motionless cirrus cloud.[18] In such cases the observations were, no doubt, made in the north and northwestern octant of a cyclone, where the cyclonic and progressive motions very nearly or quite counteract each other.

For the same reasons that cirrus clouds have almost universally an easterly motion, the winds on the tops of high mountains are mostly westerly. The large velocities of the general progressive and nearly easterly motions here are rarely entirely counteracted by cyclonic or any kind of abnormal motions of the atmosphere. In four hundred and thirty-two cases, in which the wind blew with a velocity of 65 miles or more per hour on the top of Mount Washington, Loomis found that 87 per cent. came from the W., NW., and N., while only 4 per cent. came from the NE., E., and SE.[22] From a similar examination of three hundred and sixty-three cases of high winds on the top of Pike's Peak, he found them distributed with regard to direction as follows: N., 28; NW., 47; W., 154; SW., 111; S., 18; SE., 1; E., 0; NE., 4.

Stationary cyclones.

205. If for some reason there is a local and permanent cause of temperature disturbance between the central and exterior part of any somewhat circular portion of the atmosphere, we have roughly the conditions of a cyclone which is independent of a state of unstable equilibrium, but which would be aided and strengthened by such a state. The cause of the temperature disturbance being fixed, of course the cyclone cannot have a progressive motion.

The conditions of such a cyclone are found in the northern part of the North Atlantic Ocean, in which there is a considerable area in which the temperature is higher than that of the surrounding parts, and this is especially the case in the winter season. This is seen from the abnormals of mean temperature given in § 119. By locating these on a map of the North Atlantic, by means of their latitudes and longitudes, it will be found that the central part of them, including mostly the largest ones, is located a little northeast from Iceland, and the abnormal temperature above the normal temperature of the latitude is about 16° F., the reason of which is given in § 114. In January, however, the temperature on the continents in those latitudes has become very much less, while that of the ocean has changed but little, so that at that time the abnormals of temperature are at least twice as great, and we have the conditions of a cyclone of twice as much violence. But in July the reverse takes place; there is an equalizing somewhat of the temperatures in all longitudes, and the abnormals are very small and change signs, the region about Iceland then being a little below the normal of latitude for that month.

Similar conditions are found in the northern part of the North Pacific Ocean, but not so marked. The greatest abnormal temperature of the mean of the year being about 8° F., and that for January more than twice as great.

Hence, in the extreme northern parts of both the Atlantic and Pacific Oceans, we have the conditions of a permanent cyclone, which are observed both in their effects upon the winds and the barometric pressures of these regions, especially in the winter season. The average winds across the Atlantic in middle latitudes and south of Iceland are from the west and far above the average strength of the latitude, adjacent to Great Britain and along the coast of Norway from the southwest, and curving around toward the west in the extreme northern part of the ocean, in Greenland and adjacent to it they are from the northeast. This cyclonic motion causes a mean barometric depression below the normal mean pressure of the latitude of about 4^{mm}, but for January this becomes more than twice as great, since the abnormals of temperature for that month are greater in about the same proportion than they are for the mean of the year, and then the prevailing west winds across the Atlantic in middle latitudes are unusually strong.

In the summer season the temperature of that region is so nearly the same as the normal temperature of latitude that there is no observable cyclonic action or diminution of pressure below the normal pressure of the latitude. The gentle west winds then in the middle latitudes of the North Atlantic are simply those of the general motions of the atmosphere belonging to those latitudes. In the northern part of the North Pacific there is observed, especially in the winter season, a similar cyclonic action and a diminution of barometric pressure in the central part.

This great stationary cyclone, being a local disturbance of the general motions of the atmosphere, has a considerable influence on the progressive directions of cyclones crossing the Atlantic from America to Europe. As these approach Europe they are drawn around on the southeast side of this cyclone from a nearly easterly direction towards the northeast, and pass mostly up toward Norway, so that of the few which cross over in middle latitudes from America scarcely any enter Great Britain or France.

At the same times that the northern part of each ocean has a temperature with the central part considerably above the normal of latitude, there is a large area in the northern part of each continent with a central temperature below this normal. This gives roughly the condition of a stationary cyclone with a cold center, but as the areas are large and include a land surface with mountain ranges and high table lands, there is perhaps but little cyclonic effect produced, for over so large areas the whole depth of atmosphere may be regarded as only a very thin disk in which the friction of cyclonic motions would be very great in comparison with the forces. This is exemplified in the case of the general motions of the atmosphere for each hemisphere, which are simply two great hemispherical cyclonic systems of cyclone and anticyclone, with cold central parts in the polar regions. The great amount of temperature disturbance in these cases, and the greater deflective effect of the earth's rotation, being as n to $n \cos \Psi$, causes a strong cyclonic motion around the south pole, manifested in the strong west winds of the middle and high southern latitudes, and also a barometric depression of about 1 inch in the polar region; but in the northern hemisphere the cyclonic action is so diminished by the greater surface friction of a mostly land surface, that the normal winds of the middle latitudes are very gentle and the barometric depression around the north pole is very small. If in addition to this the temperature disturbance were less, and also the effect of the earth's rotation, as in the case of the cyclonic conditions of Asia and North America, the cyclonic effect would be unobservable.

Areas of high barometric pressure.

206. If only a single regular cyclone, without any other abnormal disturbances, existed in any part of the globe in which the atmosphere, in its normal condition, had a uniform barometric pressure, we have seen that there must necessarily be an annulus of barometric pressure around the central part of low barometer a little above the normal or mean pressure. But it has been shown that the normal barometric pressure is disturbed and made irregular, not only by the general motions of the atmosphere, but likewise by stationary cyclones which disturb the uniformity of pressure, and give rise to gradients in an east-and-west direction. If, therefore, the inequalities of pressure of a regular cyclone are superimposed upon all these other irregularities, a chart of the resultant pressures does not give regular circular isobars, and indicate a regular annulus of high pressure, but the former are much distorted and the latter is broken up into areas of higher and lower pressure, the former being areas of high pressure. As an illustration of this we may suppose a circular wall of uniform height to be built on uneven land. This, referred to a level plain, would have higher and lower places, and would not be represented by a circular wall of the same height all around.

The annulus of high barometer of the stationary cyclone of the North Atlantic, falling on its south side upon that of the general motions of the atmosphere, which, undisturbed by abnormal irregularities, would pass around the earth on the parallel of about 35°, gives rise to the area of high pressure in the Atlantic on this parallel. The effect, also, of this annulus on the southeast and east side is to cause the highest annual mean barometer to be, not on the parallel of 35° in Europe and Asia, but considerably farther north, being, in the interior of Asia, about the parallel of 50°. In the case of any large progressive cyclone, if its annulus of high barometer falls upon this normal annulus of the parallel of 35°, or upon any high ridge or part in the various irregularities, it gives rise to an area of high barometer.

If likewise the center of a large temporary and progressive cyclone is so situated on a high latitude that its annulus of high barometer falls on that of the general motions of the atmosphere on the parallel of 30° or 35°, the resultant is a very high barometer pressure where the two coincide, and if the center is situated in the neighborhood of Lake Superior or Lake Winnipeg, the tendency is to fill up the gap in the annulus of the general motions of the atmosphere in the Mississippi Valley, where, we have seen (§ 166), it is interrupted by the curving around of the direction of the atmospheric currents from the deflection by the Rocky Mountains.

207. But areas of high barometer are most usually caused by two or more progressive cyclones interfering with and encroaching upon one another. Across the United States there is generally a pretty regular series of cyclones passing from west to east, of which the part of the annulus of high barometer on the west side of the one which precedes, falls somewhat upon that of the east side of the one which follows, causing a sort of ridge between, or if this is still interfered with by other irregularities, as it usually is, it may be an area of high barometer of almost any shape.

In forty-four cases of such ridges or areas of high barometer, with an area of low barometer between, passing over the United States, Loomis[23] found the average distance from the center of low barometer to that of the areas of high barometer preceding and following to be about 1,000 miles, and the average height of the barometer about 30.35 inches, the normal height being about 30 inches. This indicates that the average height of the annuli of highest barometer was about 0.2 inch above the normal height, supposing that the highest part of each did not in general fall exactly together.

208. Two areas of high barometer during the winter season are found in the central parts of North America and of Asia. These arise from the increased density of the air, due to the greater cold of these regions at that season, and forming imperfectly the conditions of a cyclone with a cold center; but on account of the great extent of area and its uneven surface the cyclonic motion is not sufficient to give a value to v in (7), which gives to the first term of the second member a value sufficient to overcome the next two terms of that member depending upon

friction and inertia, which requires a gradient of pressure increasing toward the center where there is no force depending upon the earth's rotation, since the radial motion in this case is not from the center. With considerable gyratory motion the gradient would be reversed, and the pressure be the least in the central part, as in the case of an ordinary cyclone, and as is the case in the general motions of the atmosphere in which the central polar regions are the coldest.

Since the pressure in the central part depends upon extent of area, the pressure in the interior of Asia is much greater than that in the interior of North America. While, therefore, the barometric pressure of the former in January is only about 4^{mm} above the annual mean, that of the latter is more than three times as great.

209. As the atmosphere over an area of high barometer gradually settles down in the interior and flows out from beneath on all sides, and in from all sides above to supply its place, there can be no condensation of vapor even if the air were saturated, and the interior being supplied with air from above, it must soon be occupied with very dry air, since there is not only very little vapor in the cool air above, but the capacity of the air for moisture is being continually increased as its temperature is increased from compression and for other reasons in coming down toward the earth's surface. While, therefore, areas of low barometer are usually areas of cloud and rain, § 203, those of high barometer are usually areas of clear sky and dry atmosphere. If, however, the whole atmosphere over a large area is nearly saturated with vapor and an area of high barometer should be caused by two or more surrounding cyclones, local conditions may give rise to ascending currents, and consequently to cloud and rain, where the barometric pressure is considerably above the mean normal pressure of the region.

Since the atmosphere of an area of high barometer is usually very dry and clear, especially in high latitudes in the winter season, it allows the dark radiations of the earth's surface and of the contiguous air to escape through the atmosphere above into space with much greater facility than usual, so that there is a rapid cooling of the earth's surface and lowering of the air temperature soon after the passing away of an area of low pressure followed by one of high pressure and clear dry atmosphere. To this is to be attributed, in a great measure, the usual decrease of temperature in clearing weather on the west side of a cyclone. This effect is also increased to some extent on the earth's surface by the greater amount of evaporation on that side, since the air is drier and the surface of the earth damper on account of the rain which usually falls during the passage of the cyclone.

Since the dry atmosphere has comparatively little radiating power, the cooling takes place mostly on the earth's surface, the contiguous air being cooled mostly from contact with the earth's surface; hence the cooling is so much greater below than at some altitude above the earth's surface that the vertical temperature gradient after a long continuance of high barometer often becomes inverted, and the temperature increases, instead of decreasing, with increase of altitude. This is especially the case in the interior of Asia during the winter, in the almost total absence of any heating effect from the sun's rays. The whole surface of the earth becomes cooled down to a very low temperature, and where there are hills and valleys the cold air runs down from the hill-sides into the valleys and accumulates there. The surface temperature becomes especially low where there is a covering of snow, since this, being a poor conductor of heat, does not allow the heat in store beneath the earth's surface to come through readily and heat the air above.

210. In lower latitudes, after a continuance for some time of an unusually great barometric pressure in winter, the vertical temperature often becomes inverted, and continues so for one or two weeks. A remarkable example of this kind occurred in France during two periods of January, 1880. The following table shows the minimum temperatures observed during these periods at the several places named:[24]

Dates.	Parc St. Maur, Paris (altitude 46^m).	Poitiers (altitude 117^m).	Clermont (altitude 407^m).	Puy-de-Dôme (altitude $1,467^m$).	Pic du Midi (altitude $2,366^m$).
	°	°	°	°	°
4	−1.4	−0.4	−4.0	−3.0	0.2
5	−1.5	−0.6	−6.0	1.0	0.0
6	−1.1	−0.3	−4.0	−2.0	−3.5
7	−1.5	−0.0	−1.4	−1.0	−3.3
8	−3.5	−2.3	−7.0	−1.0	−4.0
9	−2.0	−3.3	−7.0	−2.0	−4.2
10	−3.4	−4.4	−7.0	−2.0	−5.0
11	−0.7	−4.2	−8.0	−1.0	−4.5
12	−4.8	−6.1	−12.0	−2.0	−5.2
13	−7.6	−5.4	−7.0	0.0	−5.4
14	Wanting.	−3.2	−10.0	−6.0	−4.8
25	−7.5	−8.1	−14.0	−7.0	−10.9
26	−9.8	−7.1	−10.0	−5.0	−8.0
27	−9.0	−6.1	−12.0	−2.0	−8.2
28	−11.5	−8.0	−14.0	−1.0	−8.2
29	−11.0	−5.9	−5.0	−1.0	−5.8
30	−5.3	−0.4	−4.0	1.0	−5.2
31	−5.4	−1.7	−4.0	1.0	−5.2

The Puy-dé-Dôme is about 10 kilometers from Clermont.

From this table it is seen that during the first and the last parts of the month the temperature of the air at Paris, Poitiers, and Clermont, was generally less than on the top of the Puy-de-Dôme, and frequently less than that on the top of the Pic du Midi.

According to the late Professor Plantamour,[23] "it happens every year that the temperature on St. Bernard at several hours, or even during several days, of December is higher than at Geneva. But during

December of 1879, this anomaly lasted during a longer period of time than usual. The average temperature of St. Bernard was 8°.4 C. higher than at Geneva. Out of the thirty-one days of the month, only during fourteen days was it from 0°.4 to 6°.2 C. lower than at Geneva, while during the seventeen days it exceeded this by 2° to 16°.4." At such times the vapor of the motionless air in the valleys and depressions amongst the Alps is partially condensed into fog in all the strata nearest to the earth's surface up to the altitude where the temperature of the air stratum becomes less than that of the dew-point; and as this is very nearly or quite a horizontal stratum, such valleys and depressions seem then to be filled with a sea of a milk-white surface.

211. Since areas of high barometer are mostly caused by the gyrations of cyclones or any kind of irregular gyrations of the air, around about these areas, at some considerable altitude above the earth's surface where these gyrations are not much retarded by friction, and the air is left comparatively free to flow out near the earth's surface from beneath the accumulation—if the area of high barometer is situated in the middle or higher latitudes—this gives rise to the flow towards the equator of a thin stratum of cold air near the earth's surface, under air which is comparatively much warmer and generally more moist; and as warm and cold air do not readily intermingle, this thin, cold stratum may flow to a long distance and still retain its contrast in temperature with the air under which it flows. In such cases, where the strata of different temperatures are in contact, which is usually at a very moderate elevation, there is a stratum of thin cloud formed which in some cases, if the air above is very moist, is thick enough and dense enough to give rise to a mist or a little rain of very fine particles. When such areas occur in the northern part of the United States, and the air in the middle latitudes is very moist, there is then usually a cool N.NE. wind and damp drizzly weather for several days, and especially if this occurs in mild weather in the winter season. In dry weather this usually gives rise to a sudden change of temperature, without rain and little cloud formation, and is called a "cool wave." At the first approach of this wave a thin film of cloud is often formed at a moderate altitude, where it is in contact with the warmer and moister air above, which may continue some time and appear threatening, but such clouds usually pass away without rain and often in a very short time.

In the summer season, when the intervening surface of the earth is warm and dry, we are not safe in predicting a cool wave in the middle and southern parts of the United States from the occurrence of an area of high barometer and low temperature in the distant Northwest, since the thin stratum of this "cool wave" becomes warmed up before flowing very far, and before it reaches us. But in the winter season, when the earth's surface is usually cold and wet, and especially if the ground is covered with snow, which prevents the heat of the warmer strata below from coming through readily to warm up the air above, the flow of cold air from the northwest or north extends to a long distance before its temperature is much changed, and gives rise to unusually cold weather in lower latitudes. The coldest winter weather is experienced in the middle or southern latitudes of the United States when the whole country is covered with snow, and there is an area of very high barometer and low temperature in the northern part or in British America. If these areas pass eastward very slowly, or there is a quick succession of them, the cold spell continues a long time.

Effect of cyclones on isotherms and isobars.

212. Without cyclones, either temporary and progressive or of long duration and stationary, or atmospheric disturbances of any kind arising from variations of temperature from the normal of latitude, we should have the temperature, pressure, and velocity and direction of wind at all places on the globe corresponding to those of the general motions of the atmosphere treated in Chapter III. In case of a perfectly homogeneous surface these would be the same on the same parallel of latitude all around the globe, but in the real case of nature they are subject to some variations on account of the gyrations caused by deflections from continents and mountain ranges (§ 171). In the general motions of the atmosphere the normal temperatures of latitude and the temperature gradients arising from variations of these normal temperatures with latitude have already been taken into account, and in the theory of cyclones and other disturbances we are only concerned with deviations of temperature from the normals of latitude, the resulting motions and changes of pressure being very nearly though not quite the same as if these temperature variations occurred on a globe with a quiescent atmosphere and uniform temperature.

In the treatment of cyclones the assumed regular initial gradients of temperature, increasing or decreasing in the same manner in all directions from the center outward, are not absolute gradients, but simply gradients relatively to the normal temperatures of latitude, including variations arising from the deflections of continents and from large permanent cyclones within the limits of which they may occur. In like manner the resulting gradients of pressure and the velocities and directions of the wind are not absolute ones, but simply such, or very nearly, as would result from a similar temperature disturbance in a quiescent atmosphere of uniform temperature. In order to obtain the absolute initial isotherms we must add the local variation of temperature giving rise to the cyclone to the normal undisturbed temperatures. In order also to obtain absolute pressure gradients and wind velocities and directions, we must combine the parts due to the cyclonic actions with those belonging to the general motions of the atmosphere.

213. The action of a cyclone in an atmosphere without normal gradients of temperature depending upon variations of temperature with latitude, would not change the relations of temperature and of tempera-

ture gradients to the center of cyclonic action; but this is not so where the normal undisturbed temperature varies with latitude, as it generally does. In this case the colder air of a higher latitude is being continually whirled around to the west side, and the warmer air of a lower latitude to the east side. The tendency of this action is to cause a gradient of temperature decreasing from east to west, and this, combined with the normal gradient of temperature decreasing toward the pole, gives a gradient of temperature decreasing in some northwesterly direction in the northern hemisphere and southwesterly in the southern hemisphere.

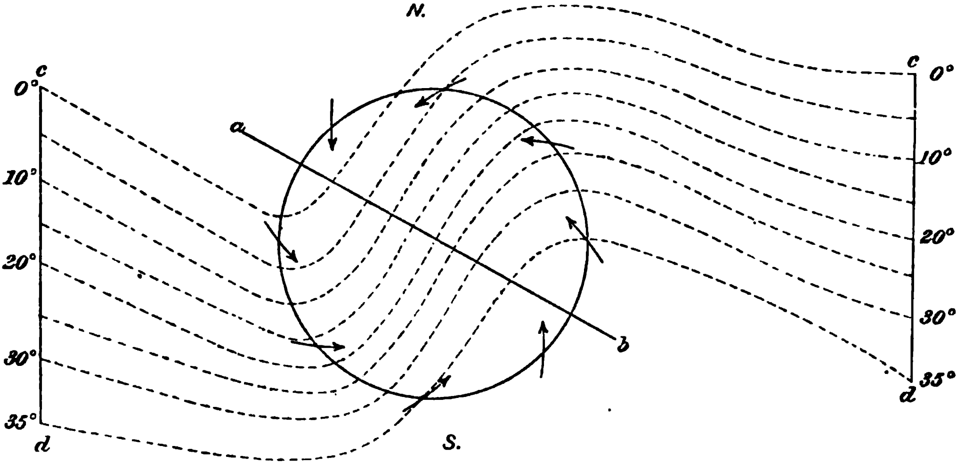


FIG. 5.

The effect upon the absolute isotherms of the northern hemisphere is somewhat as represented in Fig. 5. The isotherms which, before cyclonic disturbance, are perhaps east and west or very nearly, are thrown up on the east side and down on the west and the temperature gradient in the direction of the line *a b*, since it is the resultant of two gradients, may be much steeper than the normal gradient from north to south in the direction of *c d* before the cyclonic disturbance. In the central part of the cyclone, therefore, the isotherms extend in a northeasterly and southwesterly direction, or it may be in a direction still more nearly north and south, and there is then a crowding together there of the isotherms. There is, therefore, in a central zone of the cyclone, at right angles to the line *a b*, strong contrasts of temperature on the two sides and contrary directions of the wind.

These deductions from theory are verified by the weather charts of the Signal Service in the case of any well-developed cyclone, and especially by Finley's Preliminary Tornado Charts for February 19, March 11 and 25, and April 1, 1884. Of course the observed effects are not so regular, since we never have a perfectly regular cyclone, as assumed above, and there are always other abnormal disturbances of temperature combined with the cyclonic.

For the reasons just given, and also on account of the greater radiation and evaporation on the west and clearing side of a cyclone (§ 209), the temperature is often very much less on the west than on the east side, and it is readily seen that in its easterly progression there must be a fall of temperature during its passage over any place. This must be greater or less in proportion to the violence of the cyclone and the normal temperature gradient before cyclonic disturbance, depending upon difference of latitude. This gradient is much less in summer than in winter, since in the former season the decrease of temperature with increase of latitude is comparatively small, and hence such changes of temperature are usually much less in summer than in winter.

Great changes of temperature, however, often take place which cannot be accounted for in this way, especially in winter, which seem to arise from the greater terrestrial radiation of a particularly clear atmosphere at times generally of high barometer, the heat received from the sun being but little increased from this cause at the season when the whole amount received is small.

The effect of this cyclonic action upon the difference of temperature between the absolute and the normal undisturbed temperature, upon which alone the action of the cyclone depends, is to continually increase it on the east side, and thus to throw the center of abnormal temperature disturbance a little in advance of the center of the cyclone, and thus to continually cause a new center of temperature disturbance and the formation of a new cyclone a little in advance of the old one by which the progressive easterly motion becomes a little greater than that of the general motions of the atmosphere, as has been already explained in § 199.

214. Corresponding to the gradients of the normal temperatures of latitude, there are also gradients of normal pressure of latitude with corresponding wind velocities and directions. Without these the isobars of a regular cyclone would be circular, and the velocities and directions of the winds would be symmetrical on all sides. But the observed isobars are usually not such, since the absolute observed pressures depend not only upon the action of the cyclone, but likewise upon the general motions of the atmosphere, and it may be upon other large, permanent, and stationary cyclones.

In general the gradients arising from all the latter combined are small in comparison with those of the cyclone, and tend merely to distort them more or less; so that instead of isobars running east and west, or nearly, before cyclonic disturbance, we have isobars returning into themselves, but more or less distorted from a circle in proportion to the magnitude of the gradients with which the cyclonic gradient is combined to give the observed resultant. But in the middle and higher latitudes of the southern hemisphere, as in the region of Cape

Horn, where the gradients of the normal pressure of latitude, depending upon the general motions of the atmosphere, are greater than those of many cyclones of considerable violence, the isobars in such cyclones would not return into themselves, but the effect would be simply to cause great deviations in the normal isobars extending around the earth from the same parallel of latitude, and a crowding of them together on the equatorial side in the vicinity of the cyclone, and weather charts for this region would give isobars returning into themselves in the cases only of very great cyclonic disturbances.

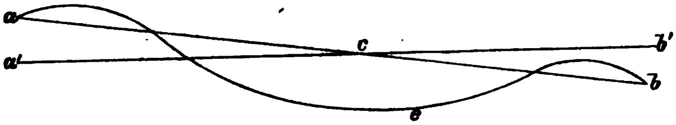


FIG. 6.

In Fig. 6 let the line *a b* represent the normal barometric gradient, depending upon the general motions of the atmosphere, combined with those of any large stationary cyclones, if there are such at the time, and the curved line *a e b*, considered with reference to *a b*, the barometric disturbance of pressure in a central vertical section, arising from some local transitory cyclone with its center at *c*. The co-ordinates of the curve *a e b*, with reference to the horizontal line *a' b'*, then represents the resultant disturbance of pressure arising from both causes, and the steepness of the curve at different points represents the barometric gradient. While the barometric gradient, as represented by *a e b* with reference to *a b*, is symmetrical on the two sides of the center *c*, that of the resultant, which is represented by the same line referred to *a' b'*, is very steep on the one side, but it almost vanishes on the other, and the point of lowest pressure does not coincide at all with the center *c* of the cyclone, but is thrown off in the direction in which the normal gradient declines most rapidly. Hence, in general, the point of lowest pressure does not indicate accurately the center of the cyclone.

In the latitude of Cape Horn, where the normal gradient depending upon the general motions of the atmosphere, and represented by *a b*, is very great, it is readily seen that with only a small cyclonic barometric depression there would be no central point, *e*, of minimum pressure, but only a very much diminished gradient, which would be in the same direction at all points between *a'* and *b'*. In this case there would be no isobars returning into themselves, and the position of the center of cyclonic action would be still more uncertain.

Another example of this sort is found, especially in the winter season, in the North Atlantic Ocean. The great stationary cyclone prevailing here at that season, with its center near Iceland, gives rise to a considerable gradient of barometric pressure increasing in all directions outward. The temporary and progressive cyclones which cross the

Atlantic in the middle latitudes are carried around on the southern and eastern sides of this cyclone often in the steepest part of the gradient. Consequently the resultant gradient of both cyclones is steep on the one side of the smaller progressive cyclone and the isobars close, while on the other side the gradient is small and the isobars comparatively wide apart. This is exemplified in the case of the great storm of February 5, 1870, of which Loomis[27] has given a chart of the isobars. If the barometric depression of the smaller progressive cyclone had been small, the isobars would not have come around on the one side so as to form an inclosure.

IV.—RELATION BETWEEN THE BAROMETRIC GRADIENT AND THE VELOCITY OF THE WIND.

The relations heretofore obtained.

215. Equation (53), Chap. III, is an expression of the gradient of the normal pressure of latitude in the case of the general motions of the atmosphere. This same expression, for reasons already given (§ 185), is applicable in the case of a cyclone if we put cos $\theta=1$ and n cos ψ for n. With these changes it becomes

$$(a) \qquad G=\frac{1077.4\,(2n\cos\psi+\nu)\,s}{\cos i\,(1+.004\tau)}\cdot\frac{P}{P_0}$$

in which G is expressed in millimeters and not in meters, and in which

$$(b) \qquad \nu=\frac{v}{\rho}=s\cos i$$

The expression is the same as in (5_2), Chap. III, using the modified notation (§ 185) in this case, ρ here being the linear distance from the center since the angular distance θ is supposed to be so small that sin θ does not differ sensibly from the arc.

In the general motions of the atmosphere ν is always so small in comparison with $2n$ that it can be neglected in comparison. This, however, is not the case in cyclones, especially near the center, where ρ in the expression (b) above becomes small. For instance, on the parallel of 45° where cos $\theta=0.7071$, we have by Table XIV, where the hour is the unit of time,

$$2n\cos\psi=2\times.00007292\times3600\times0.7071=0.3714$$

In a cyclone with a linear gyratory velocity v of 50^{km} (31 miles) (b) above gives at the distance ρ from the center of 500^{km} $\nu=0.1$, which is more than one-fourth part of the value of $2n\cos\psi$ above. But if, under the same circumstances the distance ρ from the center were only 50^{km} we should have $\nu=1$, and it would then become the principal term in $(2n\cos\psi+\nu)$ in the expression of G in (a). Putting for s in (a) its value deduced from (b), we get the factor $\rho\nu^2$ in the expression of G; but $\rho\nu^2$ is the expression of the centrifugal force arising from the gyratory motion. The part, therefore, of the gradient depending upon the centrifugal force of gyration is the part of (a) depending upon ν, while the other part, that depending upon n, is the part depending upon the influence of the earth's rotation.

It is seen from (a) that the value of G, for any given velocity of the wind s, depends also upon the values of P and of i, and in a small measure also upon the temperature τ. The less the value of P and the greater the value of τ the less the density of the air, and consequently the less the force depending upon the earth's rotation and the centrifugal force which causes the gradient. The greater, also, the inclination i is the smaller is $\cos i$ and the greater is the gradient G, all other circumstances remaining the same.

216. From (a) and (b) we get

$$(c) \qquad s^2+as=b\ G$$

in which

$$(d) \qquad a=\frac{.00014585\ \rho\cos\Psi}{\cos i} \qquad\qquad b=\rho\ (1+.004\tau)\frac{P}{P_0}$$

But by the formula for the solution of quadratic equations we have

$$(e) \qquad s=-\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2+bG}$$

By means of this and (d) we get the velocity s of the wind in a cyclone when the values of G, ρ, φ, i, P, and τ are known. Where the altitude is given instead of P, the latter can be obtained with sufficient accuracy from the table of § 27, chap. 1. At and near sea-level we can generally put $P : Po=1$ without material error. The value of i, in any individual case, is generally unknown, and the best that can be done is to use some average value for the latitude of the place and the kind of surface, obtained from the discussion of observations. As its value depends upon friction it is usually less upon sea than upon land. The further also from the center of the cyclone the greater its value, as deduced from theory and corroborated by observation (§ 196).

In the preceding formulæ ρ and s must be expressed in meters per second, and G in millimeters of the mercurial column for one degree of a great circle.

When the unit of distance is 1 kilometer, and the unit of time one hour, we must put

$$(f) \qquad a=\frac{0.52505\ \rho\cos\psi}{\cos i} \qquad\qquad b=\frac{\rho\,(1+.004\tau)}{.083055}\cdot\frac{P}{P_0}$$

For the unit of distance 1 mile and the unit of time one hour, and the gradient in inches per 60 geographic miles,

$$(g) \qquad a=0.52505\ \rho\cos\psi \qquad\qquad b=\frac{\rho\,(1+.004\,\tau)}{.005262}\cdot\frac{P}{P_0}$$

Corrections needed and the cause.

217. It has been clearly shown by Loomis[25] from the comparison of the average of a great many values of G, given by (a) for any observed values of s, with the average of the observed values of G corresponding with these same values of s, in very numerous cases both on land and on sea, and under nearly all the different varying circumstances, that the theoretical value of G for any given value of s, as given by (a), is always too small, and, conversely, for any given value of G, the theoretical value of s, as obtained from (c), is too large.

From an examination of one hundred and twenty-three cases taken from the Signal Service weather maps of the United States he has found that the theoretical value of G given by (a) is only about 61 per cent. of the true observed value, and from an examination of eighty-one cases taken from Hoffmeyer's charts of the North Atlantic for three years, that the theoretical value of G there is about 73 per cent. of the observed value. From a comparison, also, of the averages of the barometric gradients given by five years' observations at Kew with the computed values, and likewise in the case of the great storm of February 5, 1870, on the northern part of the North Atlantic Ocean, he obtained results differing but little from those obtained in the previous cases for land and sea, respectively. It would seem, therefore, that the error of the formula is greater on land than on sea, but this may arise from improper values in miles given to the scale estimates of the forces of the wind at sea.

The error evidently arises from the assumption in § 157 that the effect of friction is in the direction of the motion of the air over the earth's surface, and therefore that in (46) we have $F_u = -F_s \sin i$. The frictional resistance of any stratum of air moving over the earth's surface is both from the earth's surface and from the stratum above it. In a cyclone it has been shown that the inclination or value of i depends upon friction, and is therefore much greater at the surface than only at a small elevation above. The direction of motion, therefore, of the air of the strata above in a cyclone differs considerably from that at the surface of the earth, but it is only in the case where the motions of the air are in the same direction in all the strata that the whole resultant effect of friction upon the lower stratum, that of the earth's surface on the underside in the one direction and that of the stratum above in the contrary direction, is in the direction of motion of the air in that stratum.

Since the inclination is greatest in the lower stratum there is a radial negative component of velocity, that is, a component of velocity toward the center, relatively to that of the stratum above, and consequently a negative force required to overcome the frictional effect of the upper stratum upon the one below. The value of F_u, therefore, regarded as either positive or negative, is greater than in the expression $F_u = -F_s \sin i$ above, which enters directly into the expression of (50), and indirectly into that of (54), § 157. Since F_u and dU always have the same sign, negative in the case of an ordinary cyclone and the reverse in a cyclone with a cold center, the effect of this increased value in all cases, as seen from (50), must be to make the gradient greater than that given by (54), deduced from the value of $F_u = -F_s \sin i$.

The effect upon the value of G of this additional consideration taken into the theory can only be determined from observation, and according to Loomis's results its effect is to increase it very nearly in a constant ratio under all circumstances, at least for the same kind of surface, as of land or water. Disregarding any distinction between land and sea, it would seem that this ratio must be about 1.5, that is, that the value of G given by (a) must be increased about one-half. To obtain the value of s, therefore, from the observed gradient, by means of (c), it will be necessary to divide the last number by 1.5.

As an example of the application of (c) in a special case of large barometric gradient and velocity of the wind, we may take the great Scottish hurricane of the 24th of January, 1868[28]. From observations of the barometer at noon of Thursday, at Aberdeen and Culloden, it was found that there was a difference of 1 inch in 138 miles, giving a gradient of 0.5 inch very nearly. As this is a case in which the gyration is near the center and very rapid, and also in a high latitude, we may suppose the value of i is so small that we can put $\cos i = 1$ without sensible error. We may also put for January $\tau = 0$, for a large error in temperature produces only a very small effect. The value of r is not known, but we shall put it at 160 miles. We can also put $P = P_0$. The value, also, of Ψ, the polar distance, may be put equal 32°.

Using in this example the expressions in (g), since the units are mile and hour and the gradient expressed in inches, we get, with the data above, $a = 71.24$, $b = 30473$. With these values of a and b, and the value of $G = 0.5$ inch, we get from (e) the velocity of the wind $s = 92.9$ miles. If, however, we divide $b\,G$ by 1.5, we get $s = 71.4$ miles.

Velocities corresponding to given gradients greater in winter than in summer.

218. Mr. Ley has found from the discussion of a great many observations taken from the daily telegraphic reports issued by the Meteorological Office (London), from August, 1870, to July, 1875, that "*the mean velocity of the wind corresponding to each gradient is much higher in winter than in summer.*" The stations selected were Stonyhurst and Kew. There is a remarkable conformity in the results for the two stations. The averages of these are given in the following table:

Gradient per 1°.	Winter (October-April).		Summer (April-September).	
	Number of instances.	Mean velocity.	Number of instances.	Mean velocity.
Inches.		*Miles.*		*Miles.*
0.24	142	8.25	274	6.07
0.36	245	4.90	402	8.01
0.48	317	6.75	406	11.17
0.60	305	10.75	274	14.60
0.72	191	12.96	133	16.47
0.84	170	14.15	101	18.47
0.96	149	16.98	77	20.98
1.08	93	21.12		
1.20	66	24.73		

The results in this table show very decided differences in the velocities for the same gradients in winter and summer. We have seen that the average inclination of cyclones in this region is greater in winter than in summer, § 204, and it is also evident from an inspection of (*a*) that the greater the value of i, the less is that of s, for the same gradient G. This is especially the case for small gradients which are mostly on the outer part of the cyclone where i is large. Hence the velocities for the same gradients are less in winter than in summer. A small part of this effect, however, arises from the smaller value of τ in the formula in winter than in summer. The effect is simply a local one due to the proximity of the great stationary cyclone in the northern part of the North Atlantic, which prevails mostly in winter and nearly vanishes in summer.

Examples.

1. Given at the earth's surface, $v=29.1^{m}$ per second, $\tau=12^{\circ}$ C., $i=o$, $\psi=39^{\circ}$, rho $\rho=764^{km}$, what is the value of G? (*a*) Answer. 13.7^{mm}.

2. Given at an altitude where the pressure $P=733^{mm}$, $G=3.73^{mm}$, $i=36.8^{\circ}$, $\tau=-2^{\circ}$, $\psi=45^{\circ}$, and $\rho=249^{km}$, what is the value of s? Answer. 12.2^{mm}. (*c*) and (*d*).

3. What is the value of s in the same example, using the divisor 1.5 in the last member of (*c*)?

4. Given $G=0.11$ inch, $i=25^{\circ}$, $\psi=50^{\circ}$ $\tau=15^{\circ}$ C., $\rho=170$ miles, what is the value of v in miles per hour? Answer. 37.4. (*e*) and (*g*).

5. What, in the same example, using the divisor 1.5 in the second member of (*c*)?

CHAPTER V.

TORNADOES.

Introduction.

219. Besides the general motions of the atmosphere in connection with those of cyclones, either stationary and more or less permanent, or progressive and transitory, there are various other local disturbances called tornadoes, hail-storms, waterspouts, &c., which occupy at any one time only a very small portion of the earth's surface in comparison with that occupied by a cyclone, but which, over this small area, are characterized by far greater violence and destructiveness. They are somewhat similar in their general character to one another and also to small cyclones, and all depend in some measure upon the same conditions of the atmosphere. The distinctions between them arise simply from small variations in these general conditions, and all may be included under the general head of tornadoes.

Tornadoes differ from cyclones mostly in their extent. They have the same interchanging motion of atmosphere between the central and the exterior parts, arising from the same cause, a difference of temperature, an ascent of atmosphere in the interior, a flowing out above, and a gradual descent in the outer surrounding part. The conditions, however, from which they arise are somewhat different, and this causes the difference in their magnitude. In the cyclone, as well as the tornado, the most important condition is that the atmosphere shall be in the state of unstable equilibrium. But in addition to this, a cyclone must have the condition of an initial increase of temperature, depending upon primary causes and not upon the action of the cyclone, over a considerable area, so as to determine the initial motions over a large area toward some central point. The greater this area the greater is the initial extent of the cyclone, and the gyratory motion depends upon the effect of the earth's rotation, and may be, and generally is, sensibly independent of any initial motions of the atmosphere relatively to the earth's surface. On the other hand, the condition of a *tornado* in regard to temperatue is simply that of unstable equilibrium for saturated air at the existing temperature, the other condition being that the air shall have a gyrating motion relative to some central point, arising from any cause whatever. In the unstable state the lower strata are liable to burst up through the upper ones at any point where there may be some slight predetermining cause, which is never wanting, arising from a slight local temperature or other disturbance. An upward current being once started at any point, the region of ascending current, as has been explained in the case of cyclones, is kept warmer, and consequently rarer, as long as the ascending current is supplied with air nearly or quite saturated, or until, from an inversion of the air in the lower and upper strata, the state of unstable equilibrium is changed. Without the other condition, however, of a disturbed and gyrating state of the atmosphere, the motion from all sides would be directly toward a central point, without the gyratory motion and violence which characterize the tornado. The case is similar to that of water in a shallow basin running out through a hole in the center. If the initial state of the water is that of perfect rest the water flows directly toward the center, with a very slow velocity, but if there is the least initial disturbance of a gyratory character when the water first begins to flow, it soon runs into rapid gyrations around that center. The case is somewhat the same in a tornado, except that instead of running *down*, the air of the lower strata runs *up* through the strata above at the place where it receives its first upward impulse.

While the conditions of a cyclone may extend over a large area and cause a correspondingly large atmospheric disturbance, those of a tornado cause only a local disturbance over a comparatively small area. Hence tornadoes are of small extent in comparison with cyclones, and although somewhat similar in other respects, they yet form a distinct class of atmospheric disturbances, and it cannot be said that there is a connecting link, or, in other words, that the smaller cyclones commence where the larger tornadoes leave off. Since the gyratory motion depends upon the influence of the earth's rotation, and this is not sensible in small areas, cyclones, at least such as have a very sensible gyratory motion, must extend over a considerable area, as observed, § 207. But in six hundred tornadoes observed in the United States, according to Finley,[27] "the width of path of destruction, supposed to measure the distance between the areas of sensible winds on the north and south side of the storm's center, varied from forty to ten thousand feet, the average being 1,085 feet."

The fundamental equations and their solution.

220. The fundamental equations of a tornado are the same as those of a cyclone (2), § 185, but in the case of a tornado the term $n \cos \psi$ can generally be omitted in comparison with ν, since the former, on account of the smallness of extent of the tornado, is supposed to have no sensible influence. Since $2n \cos \psi$ is the gyratory velocity at unit distance of the earth's surface around the center of the tornado, and ν is that of the air relatively to the earth's surface, these two terms are to each other as the absolute linear velocities at the same distance ρ from the center. For

the former of these we have in miles per hour, or 3,600 seconds, by Table XIV,

$$n\rho\cos\psi = .00007291 \times 3600\,\rho\cos\psi = 2625\,\rho\cos\psi\ .$$

This, for the extreme distance of 1 mile from the center, gives, on the parallel of 45°, where $\cos\psi = 707$, only 0.18 of a mile per hour for the linear gyratory velocity due to the earth's rotation of any point of the earth's surface around the center at the distance of 1 mile. For smaller distances from the center it is less in proportion. Hence in the case of all ordinary tornadoes, in which the distance of the gyrating air is much less than a mile, the linear gyratory velocity per hour is a very small fraction of a mile, and hence may be neglected in comparison with the usual velocities in a tornado, which often surpass 100 miles per hour. All the results, therefore, obtained in the preceding chapter from the solution of these equations in case of no friction, so far as they took into account all the terms, are applicable in the case of tornadoes, but in this case the quantity ν', which was omitted in the initial conditions of cyclones, becomes the principal determining cause of the direction of the gyrations, and cannot be neglected. We therefore get from (8) and (9), § 187, in this case, by neglecting in the former $n\cos\psi$ in comparison with ν, as it has been shown we can without sensible error, for all altitudes

(1) $$\rho^2\nu = \rho v = C$$

in which

(2) $$C = \frac{\int \rho^2 (n\cos\psi + \nu')\, dm}{m}$$

retaining $n\cos\psi$ in comparison with ν', though even here the former is very small in comparison with the latter. The effect of friction in tornadoes is very small in comparison with what it is in cyclones. The latter are very broad disks of revolving air having diameters many times greater than their depth, and hence the gyrations are very much hindered and their velocities diminished by the friction between the thin revolving disks and the earth's surface. Tornadoes, on the other hand, are rather columns of air having generally small bases in comparison with their altitudes, so that the friction is very small except at the base of the column near the earth's surface. A solution, therefore, of the equations above with the friction terms omitted, must give approximate results, especially for the upper strata, but in these as in the flowing of liquids and the spouting of fluids, some allowance must be made for the effect of friction, and the observed effects must always fall a little short of the theoretical results.

If the sum of the moments of initial gyration, expressed by the numerator of the expression of C in (2), and consequently C, equals 0, then by (1) we must have $v = 0$ for all values of ρ, and consequently there is no gyratory motion. The gyrations in this case, therefore, depend mostly upon an initial state of disturbance such as will give C a value, but not necessarily upon any initial regular gyratory motion of the air subject to tornadic action. Hence, tornadoes usually occur in a cyclone, where the necessary initial disturbances exist, and as the cyclone in the northern hemisphere gyrates from right to left, these initial disturbances generally tend to give a positive value to C, and consequently to v, since by our notation v is positive in that direction. If the initial local disturbances were such from any cause as to give C a negative value, then the gyrations would be in the contrary direction. The effect of the term $n \cos \psi$ in (2), though very small in tornadoes, is always in the positive direction, and the initial disturbances, together with the effect of this term, seem to determine the direction of gyratory velocity in tornadoes, always in the same direction. In the cases of six hundred tornadoes, according to Finley,[27] the observed gyrations were in every case from right to left.

221. Since the equations for a tornado are the same as for a cyclone given in (2) § 185, and the solution of these, in the case of no friction, is the same as in the case of those for the general motions of the atmosphere, § 149, except in the two terms which vanish in both cases, we get from (25), § 149, in the case of tornadoes, for a stratum of equal pressure, in which case the first member of that equation vanishes

$$X_0 - X = \frac{s_0^2 - s^2}{2g}, \tag{3}$$

in which X_0 and s_0 are the altitude and velocity respectively, of some assumed point regarded as the initial point of integration. This expression gives us the relation between the differences of altitude and of velocity for different points in the stratum of equal pressure, but we learn nothing from it with regard to the directions of motion. The solution also from which it has been deduced does not take into account any differences of temperature, since α, which is a function of the temperature (4), § 142, is treated as a constant. The disturbing force is supposed to be simply an initial impulse toward the center, or such as would arise from a very small difference of temperature, such that α may be regarded as a constant in the final expressions. But in this latter case we must suppose also that there is sufficient friction to prevent acceleration of motion, and the initial velocity s_0 must be the velocity at the initial point after acceleration has ceased. Without this the tendency of any difference of temperature between the interior and exterior parts would be to continually accelerate the radial interchanging motion between them. As has been stated, we learn nothing from the solution with regard to directions of motion, and consequently with regard to the relations between u, v, and x, upon which the directions of motion depend.

If in a tornado we assume that the radial velocity u and the vertical velocity $\dot{x}$ are small in comparison with the gyratory velocity v, we may then put in (3) without much error $s^2 = v^2$ and $s_0^2 = v_0^2$, for with

u and x small in comparison with v, it is seen from (26), § 149, since the squares of the velocities enter into the expression, s^2 differs but little from v^2. If, therefore, in (3) we put l for $X_0 - X$, and v and v_0 for s and s_0, respectively, we get

$$l = \frac{v_0^2 - v^2}{2g}, \tag{4}$$

in which l is the height of any point in a stratum of equal pressure corresponding to v above any horizontal plane intersecting that stratum where the gyratory velocity is v_0.

If in (1) we put ρ_0 for the value of ρ where $v = v_0$, we get

$$v = \frac{\rho_0 v_0}{\rho}. \tag{5}$$

This expression shows the relation between v and ρ, and it is seen that with only a very small value of v_0 at a considerable distance ρ_0 from the center the value of v near the center becomes enormously great, and as ρ vanishes it approximates to infinity. With this value of v (4) gives

$$l = \frac{v_0^2}{2g}\left(1 - \frac{\rho_0^2}{\rho^2}\right) \tag{6}$$

This expression shows the relation between l and ρ the distance from the center, and consequently l is negative for all values of ρ less than ρ_0 and *vice versa*.

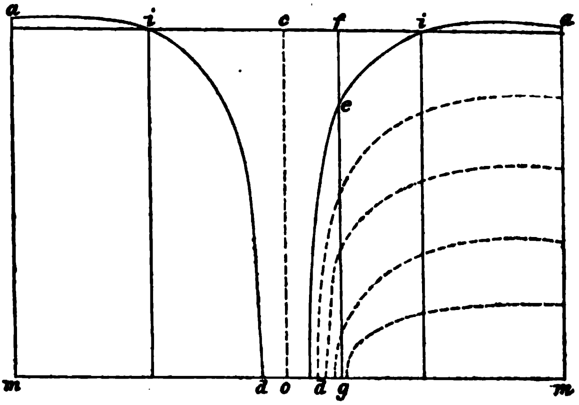


FIG. 1.

In Fig. 1 let the line $m\,a$ represent the height of an undisturbed horizontal stratum of equal pressure before being brought down in the central part by tornadic action in the form of $a\,i\,e\,d$, and let c or o be the center. Also let $f\,e$ represent the depression of the stratum of equal pressure below the undisturbed horizontal stratum $i\,i$ which intersects the line $a\,i\,e\,d$ at i, at the distance ρ_0 from the center. We shall then have $c\,i = \rho_0$, where $v = v_0$, $c\,f = \rho$, and $f\,e = l$ in (6)

If we now put

h = the height of any point of the stratum of equal pressure above the earth's surface at the distance ρ from the center,

H = the height beyond the limit of disturbance represented in the figure by $a\,m$, supposed to be at the distance ρ_0 equal to infinity,

we shall then have, since unity in (6) can be neglected for all finite values of ρ,

$$h = H - \frac{v_0^2 \rho_0^2}{2g\,\rho^2} \tag{7}$$

For any given value of v_0, corresponding to an assumed distance ρ_0, the product being always the same for all values of ρ_0, the values of h corresponding to any distances ρ from the center can be computed. It is seen that for any value whatever of H the value of h must become 0 for some finite value of ρ, and hence the stratum is brought down to the earth's surface in case of no friction, in which case alone the relation of (5) holds strictly.

For other strata at greater or less altitudes the values of h for all values of ρ would differ by a constant equal to the difference of the values of H in the two cases, and the curves representing the strata of lower altitudes, after having been lowered by the action of the tornado, would be represented by the dotted lines in the figure. The higher the altitude of the undisturbed stratum the more nearly to the center it touches the earth's surface.

222. All the preceding relations between pressure and velocity are such as satisfy the fundamental equations in case of no sensible friction and difference of temperature between the interior of the tornado and the surrounding parts. Of course we must suppose that there is some difference of temperature and interchanging motion between the interior and exterior parts in order to give the relation of (5), but this may be almost infinitely small and the final gyratory velocities at different distances become a question of time simply, unless we suppose the initial to be such as to satisfy (5), and in this case we need no unstable equilibrium or differences of temperature. But where there is the least amount of friction we must have a little difference of temperature to give a force to overcome its effects. The gyratory velocity where there is friction is the same at all altitudes, and the centrifugal force arising from it where there is no sensible temperature disturbance prevents the air from pressing in toward the center, or, in other words, the centrifugal force outward is exactly equal to that of the gradient of pressure toward the center. There is, therefore, no sensible motion of

the air toward, and ascension in, the central part, and such a motion can only arise in the case of a sensibly greater temperature in the interior than in the surrounding undisturbed part of the atmosphere. When the atmosphere is in the unstable state this difference of temperature arises from the ascensional motion itself, when once started, and then this motion continues as long as this state remains.

From (3_3), § 185, with the value of α in (4), § 142, we get

$$-gl\,(1+.004\,\tau)\,d\log P=gdX+xdx$$

when the air has a vertical motion, and

$$-gl\,(1+.004\,\tau')\,d\log P=gdX$$

when it has none, the temperature of the surrounding undisturbed air being denoted by τ'. Subtracting the two members of the latter from the corresponding ones of the former, we get

$$.004\,gl\,(\tau'-\tau)\,d\log P=xdx$$

Substituting in this the value of $d\log P$, deduced from the last of the preceding ones, we get

$$\frac{.004\,g\,(\tau-\tau')}{1+.004\,\tau'}\,d\,X=xdx. \tag{8}$$

If we now put

$$\tau=\tau_0-e\,X \qquad\qquad \tau'=\tau_0-e'\,X \tag{9}$$

then e and e' express the rates of decrease of temperature with increase of altitude in the ascending air and in the surrounding air without vertical motion respectively. With these values we get from (8)

$$xdx=\frac{.004\,g\,(e'-e)}{(1+.004\,\tau_0}\,XdX \tag{10}$$

The same is true if there is not a regular rate of decrease, provided $(e'-e)$ is constant at all altitudes.

The integration of this equation from X_0 gives:

$$x^2-x_0^2=\frac{.004\,g\,(e'-e)}{(1+.004\,\tau_0)}\left(X^2-X_0^2\right) \tag{11}$$

in which x_0 is the velocity at the height X_0. At the earth's surface we have $X_0=0$ and $x_0=0$.

If $e<e'$, that is, if the decrease of temperature with increase of altitude in the interior ascending air is less than in the surrounding undisturbed air, the value of x may be either positive or negative, and this is determined alone by the initial motion which the air receives. If $e>e'$ the first member becomes negative, which is absurd, and hence in this case there is no accelerated motion, and if the air is disturbed, the tendency is to bring this initial motion to rest. This is, therefore, simply a mathematical expression of the tendencies of the unstable state of equilibrium, already explained in a general way in § 38.

223. Although the preceding expression has been legitimately deduced from the condition of (3_3), § 185, and is the one usually given in this case, yet it is not the true expression in the case of nature, for it is evident we cannot have a continually accelerated vertical velocity up to all altitudes or values of X, but above a certain altitude it must be gradually decreased and finally brought to 0.

It must be remembered that the equations referred to do not express all the conditions to be satisfied, but that the condition of continuity must likewise be satisfied, and that no solution can be admitted in which this latter condition is not also satisfied. As the air ascends it expands into a greater volume, and only such motions can be admitted as will make room for the increased volume and yet not carry the air up indefinitely far above the general level. As the air ascends it must flow off laterally on all sides, and the ascending velocity at the limit of the atmosphere, if we can assign to it an absolute limit, must vanish, and at any rate gradually diminish in the upper strata.

As the ascending air in the interior of a tornado ascends and is relieved of pressure it expands upward and laterally, but cannot expand downward. The expansive force downward causes increased pressure at the earth's surface and in the lower strata, and the reaction causes expansive force upward to be twice as great as it would be if it were free to expand in both directions. This, in the case of no gyratory motion to disturb the horizontal strata of equal pressure, would cause the air and these strata to rise above the general level until the gradients of pressure above decreasing outward would cause the volume of outward flow on all sides to equal that of the expanded ascending air in the interior. Since the action of the expanding ascending air is exactly the same in both vertical directions in the integration of (3_3) with reference to this force, in addition to that of gravity from the bottom to the top of the column, gives no increase of pressure at the bottom, except so far as it is increased by the increase of the height of the vertical column arising from the heaping up of the air in the central part where it is ascending. The integration, also, with regard to the term $D_x x$ increases the pressure of the lower strata on the earth's surface where dx is positive, that is, where the ascending velocity is accelerated; but it is the contrary with regard to the upper strata where dx is negative. And as the amount of momentum generated and destroyed exactly balance each other, the integration through the whole volume gives nothing. The pressure of the whole column of ascending air is the same as if it were at rest, and for the whole column is given by (25), § 149, by putting s and s_0 equal 0. Since, however, the column of air is higher, the value of P is increased at all altitudes, but where $e < e'$ it is also decreased from the effect of less density arising from a greater temperature.

Where the pressure in the central part is diminished by a gyratory motion, as in a tornado, these small modifications of pressure are simply in addition to this principal disturbance.

If in the case in which $e=e'$ a single impulse were given to a portion of air, such as to cause an ascending velocity at some central part, this would be continued if there was no friction to bring the air to rest. But the relations of the velocities in the different parts would have to be such as would satisfy the condition of continuity, and this could be done with an infinite number of such relations arising from different impulses.

With $e=e'$ we get from (10) x equal a constant, but this is independent of X, and we do not know its relations to X, but simply that it is a constant with regard to time. With $e<e'$ and without friction we should have a continued acceleration as long as the conditions should remain which make $e>e'$, that is, as long as the air is in a state of unstable equilibrium. *With* friction the acceleration would continue until the friction would become equal to the forces causing the acceleration, after which the motions would be uniform, or at least only change with the relation of e to e'. This must be understood with regard to a fixed and definite amount of air. In the case of the atmosphere at large the tendency would be to continually bring new portions of air into circulation, and the forces then would be mostly required to overcome the inertia of these, and in this case little or no acceleration of velocities would be produced even without friction, unless the condition of $e<e'$ should continue a long time. The ascending velocity in a tornado, therefore, depends upon friction as well as upon $(e'-e)$, and in the case of a definite portion of the atmosphere and no friction it is continually accelerated as long as $e<e'$. The expression of (11), therefore, is not the true expression of this velocity.

224. It is seen from (87), § 36, that the rate of decrease of temperature with increase of altitude in a current of rapidly ascending and unsaturated air, where the pressure is not sensibly disturbed by tornadic action, is .00979° on the parallel of 45°, and that this may be adopted for all latitudes without material error. Putting, therefore,

τ' = the temperature at the earth's surface beyond tornadic disturbance
τ = the temperature there at the altitude H

we shall have, where the thin stratum of equal pressure is not sensibly lowered by tornadic action, as at *a*, Fig. 1,

$$(12) \qquad \tau'-\tau=.00979\,H=\frac{H}{102.1}$$

Where the air is saturated the rate is much less, and varies with difference of pressure and temperature. In this case τ can be determined for any altitude; H, approximately, by means of Table XIII, Appendix.

But according to (71), § 33, the relation between τ' and τ depends upon difference of pressure alone, and if in ascending from *m* to *b*, Fig.

1, τ' is decreased to τ, it is because of a certain amount of expansion and decrease of pressure, and the same decrease of temperature would arise from the same amount of expansion, whatever the space passed through during the expansion. As the air in a tornado ascends it is also drawn in toward the center, and therefore, wherever it proceeds from the earth's surface, and by such a motion arrives anywhere at the stratum of equal pressure, *a i e d*, Fig. 1, either by an ascent nearly vertical to the part between *a* and *i*, where this stratum is not sensibly depressed by tornadic action, or by a more lateral motion to any other part of this stratum brought down to the earth's surface at *d*, the temperature is decreased from τ' to τ, in accordance with (12). In a tornado, therefore, with rapidly ascending currents, the thin stratum of equal pressure, however much lowered in the central part by tornadic action, is likewise a stratum of equal temperature, for the air which ascends in the tornado is mostly that of the earth's surface of temperature τ', drawn in below and then ascending, and must cool down to the temperature τ on arriving anywhere at this stratum.

225. For the pressure at different distances from the center in the same horizontal plane, where u and x are so small in comparison with the gyratory velocity v that the former can be neglected in (26), § 149, and v substituted for s, we get from this equation, by means of (5), since $X-X_0$ vanishes in this case,

$$(13) \qquad \log P_0 - \log P = \frac{v^2}{360940\,(1+.004\tau)} = \frac{v_0^2 \rho_0^2}{360940\,(1+.004\tau)\,\rho^2}$$

in which P_0 is sensibly the undisturbed pressure at a great distance from the center.

The diminution of pressure in the central part of the tornado given by this expression is that alone due to the gyratory velocity. The value of u, the radial velocity, is always too small to produce any sensible effect, since the gyrations, when rapid enough to produce any sensible effect upon the pressure, are so nearly circular that u is always very small. The ascending velocity, however, in extreme cases, may be so large in comparison with the gyratory that its effect may be considerable, and, if known, should be taken into account, but in general its effect is small.

From (27), § 149, we get a relation between the pressures and velocities independent of any consideration of distance from the center of the tornado. We may suppose that s_0 is assumed at a distance from the center where its value is so small that its square may be neglected, and P_0' is then the undisturbed pressure beyond the influence of the tornado. We thus get

$$(13') \qquad \log P_0 - \log P = \frac{s^2}{360940\,(1+.004\,\tau)}$$

This is sensibly the same as the preceding in the case of no friction, since then we have $s=v$.

From this expression, the relation between P_0 and P being given, we can compute the value of s.

If in (27), § 149, considering only vertical motion, in which case $s=x$, we put ΔP_0=the effect upon P_0 of the reaction of accelerated velocity, we get sensibly, where ΔP_0 is small,

$$-\Delta \log P_0 = \frac{M \Delta P_0}{P_0} = \frac{x^2 - x_0^2}{156754\,(1+.004\tau)}$$

This, for the earth's surface where $x_0^2=0$, gives, distinguishing by an accent in this case, and putting $P_0=760^{m}$,

$$\Delta P_0' = \frac{x^2}{206.2\,(1+.004\,\tau)}\,\frac{P}{P_0} \tag{14}$$

The effect is upon the value of P' at the surface, since it cannot affect P, which is above, and it increases the difference between P' and P.

The value of x to be used is the maximum value in any part of its vertical ascent. If a less value nearer the earth's surface is used, we get only the effect of acceleration so far. It is necessary to know the pressure P of the air up where this maximum occurs, since the effect is proportional to this, or to the density where τ varies. Since the temperature in the integrations of the formula was assumed to be constant, some average value of τ should be used. A considerable variation in the value of τ in the formula affects the result but little.

With a maximum value of x equal 40 meters per second up where the pressure is one-half of the normal pressure, this expression, for the earth's surface, putting $\tau=0$, gives $\Delta P_0'=3.87^{mm}$. For a much larger value of x, however, the effect would be very much greater, since it increases as the square of x.

226. The same expression holds in the case of any horizontal motion of air of velocity x, which is gradually retarded by a resisting surface at right angles to the direction, and brought to rest in this direction. In this case the value of x must be the maximum velocity; that is, the velocity in the given direction which the air has before it suffers any sensible retardation. In this case $\Delta P_0'$ represents the whole increment of barometric pressure, since the difference in the statical pressure depending upon gravity vanishes, and is the difference between the pressure on the surface of the resisting body and that of the air generally when not resisted. If the resisting body is a thin plate, it therefore represents the difference of pressure, expressed barometrically, between the pressures on the two sides, since the static pressure on the opposite side, in the case of no friction, is not affected by the motion of the air. It is, therefore, a measure of the *force of the wind*.

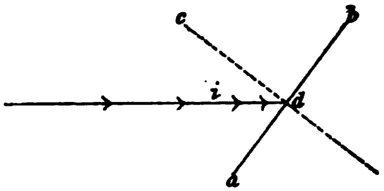


FIG. 2.

If the wind falls upon the surface *a b* of any body with an angle of incidence *i* and velocity *s* before sensibly retarded, then, instead of *x* in (14), we must put *s* cos *i*, and we get for the increase of pressure or force of the wind, at the point *d*,

$$\Delta P' = \frac{s^2 \cos^2 i}{206.2\,(1+.004\tau)} \frac{P}{P_0} \tag{15}$$

The same is deducible from (13′).

Examples.

1. With $v_0 = 3^m$ at the distance $\rho_0 = 1000^m$, what is the value of v at the distance $\rho = 25^m$? (5)

2. What is the height of the stratum at this distance, which before tornadic action was $H = 800^m$? (7)

3. What is the increase of temperature of rapidly ascending unsaturated air at the altitude of 800^m? (12)

4. What the temperature of air at this altitude with a temperature of 30° at the earth's surface and temperature of dew-point 25°? (12) and Table XIII, Appendix.

5. In the case of the first example, what was the barometric pressure at the distance of 25^m, supposing P_0 to be 760^{mm} and $\tau = 15°$? (13)

6. In the same example, what is the increase of barometric pressure in the central part from the reaction of acceleration, supposing the maximum accelerated vertical velocity is 50^m, and the pressure at the earth's surface equal 700^{mm}? (14)

7. The general pressure being $P_0 = 760^{mm}$, and that in the interior of a tornado being $P = 700^{mm}$, what is the velocity of the wind, the temperature τ being 0? (13′)

8. What, if $P_0 = 700^{mm}$, $P = 600^{mm}$, and $\tau = 20°$?

Waterspouts.

227. As soon as the ascending and expanding air in a tornado cools down to the dew-point, condensation and cloud formation take place. But this temperature is first reached in the thin stratum of equal pressure of that temperature, for we have seen that the temperature of such a stratum, whatever form it may have, is uniform. As soon, then, as the air ascends above, or enters anywhere within, this stratum, as represented by *a e d*, in Fig. 1, condensation and cloud formation take place,

and we have the phenomenon called a *waterspout*, since the part of the atmosphere sufficiently rarefied and cooled to cause condensation extends down in the form of a tapering trunk, as outlined on the two sides by $a\ i\ d$ in the figure.

If we therefore put

τ_1=the temperature of the dew-point

h_1=the height of the stratum of incipient condensation,

we then get from (7) and (12) when τ becomes τ_1

$$(16) \qquad h_1=102.1\,(\tau'-\tau_1)-\frac{v_0^2\,\rho_0^2}{2g\,\rho^2}$$

With the difference between the air temperature at the earth's surface and the temperature of the dew-point, this expression gives the altitude h_1 of the stratum of incipient cloud formation corresponding to any distance ρ from the center of a tornado, where the gyratory velocity v_0 at the distance ρ_0 is known. The first term expresses the height of this stratum around on the exterior part of the tornado where it is not sensibly depressed by tornadic action, expressed by the last term.

228. Let

P_0'=the value of P at the earth's surface and at a great distance from the center of the tornado, where P is not sensibly disturbed,

P_1=the pressure of the air when, by ascending from the earth's surface and expanding, its temperature is reduced to τ_1,

We then have from (71) § 33, in a practical form:

$$(17) \qquad \log P_0'-\log P_1=\varepsilon\Big(\log\,(273^\circ+\tau'_0)-\log\,(273^\circ+\tau_1)\Big)$$

in which, § 32, $\varepsilon=3.489$. From this the value of P_1 is obtained for any given value of τ_1. The value of τ'_0 may be regarded the same as the general value of τ' at the earth's surface, since ρ_0 is taken so as to extend to the outer part of tornadic disturbance, where neither temperature or pressure are sensibly disturbed. Within this range, however, the value of τ' is gradually decreased by rarefaction to the center of the tornado.

If we put

R= the radius of the waterspout at any altitude h_1,

V= the value of v where $\rho=R$,

we get from (16) and from (5), by putting R for ρ,

$$(18) \qquad \begin{aligned} R&=\frac{\rho_0\,v_0}{\sqrt{2\,g\,[102.1\,(\tau_0'-\tau_1)-h_1]}} \\ V&=\frac{\rho_0\,v_0}{R}=\sqrt{2\,g[102.1\,(\tau_0'-\tau_1)-h_1]} \end{aligned}$$

From these expressions the radius of a waterspout at any altitude h, and the gyrating velocity at its outer limit, may be computed when all the necessary data are given.

Denoting by R' and V' the values of R and V at the earth's surface we get by putting $h_1=0$, in (18)

$$
(19)\qquad
\begin{aligned}
R' &= \frac{\rho_0 \, v_0}{\sqrt{v_0^2+2003\,(\tau_0^1-\tau_1)}} \\
V' &= \frac{\rho_0 \, v_0}{R'} = \sqrt{v_0^2+2003\,(\tau_0'-\tau_1)}
\end{aligned}
$$

229. If with the assumed data in connection with the following figures we compute by (16) the values of the vertical co-ordinates h_1, corresponding to suitably assumed values of the abscisses ρ, or values of R by (18_1), corresponding to suitably chosen values of h_1, and represent the results graphically, we get the forms under the several circumstances as represented in these figures:

$\tau_0=30^\circ$, $\tau=25^\circ$, $v_0=1^m$.

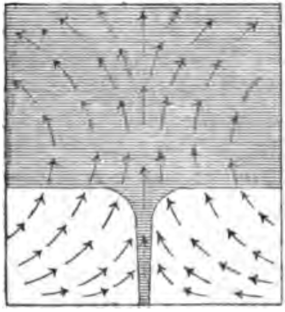


FIG. 3.

$\tau_0=30^\circ$, $\tau_1=22^\circ$, $v_0=3^m$.

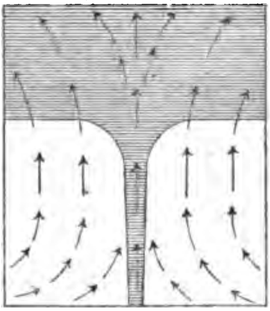


FIG. 4.

$\tau_0=30^\circ$, $\tau_1=18^\circ$, $v_0=2^m$.

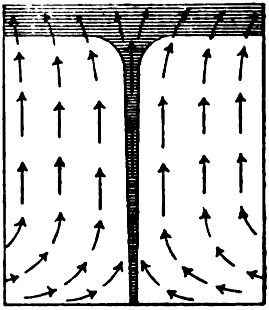


FIG. 5.

Scale: 1000 metres.

It is seen from these figures, as well as from (12), that the greater the difference between the temperature of the air at the earth's surface and that of the dew-point, the higher is the cloud and the longer the spout, and in proportion to this difference.

The values v_0 above have been assumed for the distance $\rho_0=1000$ meters. These give only a very moderate gyratory velocity at that distance, yet its increase according to the law of (5) makes it very great near the center, and the centrifugal force arising from it is sufficient to decrease the pressure very much there, and to bring down the stratum of incipient condensation to the earth's surface. In fact, however small the value of v_0, the value of P' must become infinitely small at the center, and however small the amount of aqueous vapor in the air, there would have to be a small thread-like spout of condensed vapor. This, of course, must be understood in the case only of absolutely no friction between the air and the earth's surface, or between its own strata moving with different velocities.

The directions of the resultants of the vertical and radial motions may be supposed to be somewhat as represented by the arrows in the figures, but of course these differ very much under different circumstances. But it must be borne in mind that in connection with these is the gyratory motion, which, especially near the center, is the principal one, so that while the air is ascending and inclining in toward the center, it is also gyrating rapidly around that center. As soon as the air in ascending thus comes where the tension is reduced to P_1, corresponding to the temperature τ_1, as determined by (17), condensation commences, and this determines the under side of the cloud. Near the center of the tornado the infinitely thin stratum of tension P_1, and temperature τ_1, as we have seen, is brought down to the earth's surface. Hence a water-spout is simply *the cloud brought down to the earth's surface by the rapid gyratory motion of the tornado.*

From (19_1) we get, from the three sets of conditions given with the preceding figures, R' equal 10^m, 24^m, and 13^m, respectively, and for the corresponding velocities of gyration at the outer limit of the spout at the earth's surface, we get from (19_2), V' equal 100^m, 127^m, and 156^m, respectively.

Assuming $P_0'=760^{mm}$ we get from (17), for the several sets of conditions respectively, P_1 equal 717^{mm}, 692^{mm}, and 660^m, for the tensions of the several strata of incipient condensation. For any high plateau considerably above sea-level the value of P_0' would be much less than the normal value here assumed. Of course these are the pressures also given by (13) by putting $v=V'$ or $\rho=R'$, and using for P_0 its value at the earth's surface. It is seen from this expression that where the distance ρ is very great we have sensibly $P=P_0$, and that from this there is a decrease of pressure, slowly at first, but very rapidly near the center within the spout, so that very near the center the rarefaction must be very great and the temperature extremely low, and at the center the tension of the air, upon the hypothesis of no friction, must become infinitely small and the temperature absolute zero.

In water-spouts at sea there is a rising up and foaming of the sea-water under the spout. By theory the water rises, as in a pump when relieved of pressure, 13.6 inches for each inch of barometric depression, so that if there should be a depression of 3 inches in the center of a water spout, the water would rise about 41 inches. This heaping up of the water is always observed, and on account of the rapid gyratory velocity of the air very near the center, the water is much agitated at the surface and much of it may be carried up in the interior of the spout by the ascending current. In this way the water in small ponds with fish or any other animals in it may be carried up in the air and fall at a considerable distance away. The water, however, which falls as rain at sea is always observed to be fresh water, for the small amount of sea-water carried up, when mixed with so much rain-water, is not perceptible.

Examples.

1. If $v_0=5^m$ at the distance $\rho_0=1{,}000^m$, and the temperature of the dew-point at the earth's surface is 10° below that of the air, what is the

height of the stratum of incipient condensation at the distance of $\rho=50^m$? (16)

2. What, in the same example, is the tension of the thin stratum of incipient condensation or outer limit of the water-spout, if $P'_0=760^{mm}$ and $\tau'_0=15°$? (17)

3. In the same, what is the radius of the water-spout and the gyratory velocity at the outer limit at the earth's surface? (19)

4. If $v^0=4^m$ at distance $\rho_0=1{,}000^m$, $\tau'_0=30°$, $\tau_1=15°$, and $P'_0=725^{mm}$, what are the values of P_1, R' and V' and the value of P' at the distance $\rho=30^m$?

Force of the wind and supporting power of ascending currents.

230. The velocity of the wind in a tornado being known, in order to have its force in grams per square centimeter of surface exposed normally to its direction, we must know the weight of a column of mercury with a square centimeter for its base and height equal to $\Delta P'$. For a column of pure water of standard temperature $\Delta P'$ in centimeters expresses its weight in grams, and in that case (14) would express the force of the wind in grams. In the case of mercury, therefore, (14) must be multiplied into the density of mercury, 13.596, and expressed in centimeters instead of millimeters, in order to give the pressure in grams. Hence, putting

p=the force of the wind in grams;

S=the surface in square centimeters exposed normally to the wind;

we get, using s here for the velocity of the wind, since the expression is true for any direction,

$$p=\frac{13.596\ s^2}{2062\,(1+.004\tau)}\,\frac{P}{P_0}\,S=\frac{s^2}{151.7\,(1+.004\tau)}\,\frac{P}{P_0}\,S \tag{20}$$

in which, it must be understood, P_0 is the standard pressure, 760^{mm} on the parallel of 45°. The force, therefore, is as the square of the velocity and also as the density of the air. Hence, at high altitudes, winds of the same velocities have much less force than at the earth's surface.

This expression is reduced to pounds avoirdupois per square foot, with the values of s given in miles per hour, according to Table XIV of the Appendix, by multiplying the second member by $2.04830:2.2370^2=0.40933$. We thus get in these measures

$$p=\frac{0.002698\ s^2}{1+.004\tau}\,\frac{P}{P_0}\,S \tag{21}$$

Where force is expressed in pounds, the force of a pound subject to standard gravity—that is, gravity on the parallel of 45° and at sea-level—must be understood. For other altitudes and latitudes the force of the pound would be a little different.

At sea-level, if P is not observed, we can generally put it equal to P_0 without material error. For dry air, it should be remembered, the numerical coefficient .004 in all cases becomes .003665. For ordinary hygrometric states of the air, however, the former is to be preferred, since it takes into account approximately the variations of density depending upon the most probable amount of aqueous vapor, when not observed, belonging to the observed temperature.

By (21) we can compute the force of the wind, where s is known, against any surface S exposed normally to the wind, either at sea-level or for any altitude, when P and τ are known. When the altitude is given the value of P can be obtained with sufficient accuracy from the table of § 27. With a velocity of 100 miles per hour and $\tau=0$ (21) gives, at sea-level, 27 pounds avoirdupois for the force against each square foot of surface. If the temperature were 15°, it is readily seen that this would be diminished in the ratio of 106 to 100, and at an altitude where $P=\frac{1}{2}\, P_0$ it would be still diminished one-half more. The following table gives the force of the wind in pounds upon each square foot, at sea-level, and temperature $\tau=0$, corresponding to the given velocities in miles per hour:

Velocities.	Forces.	Velocities.	Forces.
5	0.07	100	27
10	0.27	150	61
20	1.08	200	108
40	4.32	250	169
70	13.23	300	243

According to the preceding formula and table, the mechanical force of the wind in tornadoes must often be enormous. In the first of the preceding examples, § 229, although the gyratory velocity is very small at the distance of 1,000 meters, yet at the distance of 10 meters it becomes by (5) equal to 100 meters per second, or, by Table XIV, 224 miles per hour. This, it is seen from the preceding table, would give a force considerably above 100 pounds to the square foot anywhere on the earth's surface of not very great altitude above sea-level, and by (21) a force of 135 pounds at sea-level and temperature $\tau=0$.

The barometric pressure in a tornado has been observed to be nearly 3 inches below the general pressure. Supposing this latter to be 30 inches and the pressure in the tornado to be 27 inches, and temperature $\tau=25°$, we get from (13′) the value of s equal to 144 meters per second, or 323 miles per hour. This, according to (21), or the preceding table, would give a very great force to the wind. From this example we therefore see that the wind forces in a tornado must often be enormous, since the barometric depression in a tornado may be very much more than 3 inches, for observations are never made in the centers of violent tornadoes.

231. The explosive force of confined air in a tornado may be very great, and it is to this rather than to the force of the wind often that the destruction of buildings is due. If air of the ordinary tension of 30 inches is confined within a building so that it cannot escape readily, and the center of a tornado comes suddenly over it in which the tension is reduced to 27 inches, then the difference of tensions at the time is equal to one-tenth of the pressure of the atmosphere, which by (54), § 18, is 14.696 pounds to the square inch. The explosive force in this case, therefore, would be one-tenth of this, or about 211 pounds to the square foot. But the example which we have assumed is by no means an extreme one, for the tension of the air in a tornado may be diminished twice as much as this, or even very much more, and the explosive force of confined air increased accordingly. Before the confined air in the building has time to escape, the explosive force bursts asunder the walls, and throws the roof up into the air and to a considerable distance away, and the building is a wreck. If the whole roof is not carried away, at least the tin or copper sheathing is often removed, since the explosive power of the closely confined air beneath it and the material to which it is fastened is sufficient to loosen it, and it is then carried away to a great distance often by the wind. Under such circumstances, also, cellar doors have been known to be blown away from their fastenings against the force of a strong wind blowing directly against them, and also the corks of empty bottles to be blown out from the sudden expansion of the air within them.

232. Where the opposing surface is not normal to the direction of the wind, the expression of (21) is modified in the manner of (15). In the case, therefore, in which the wind blows on an irregular, or any surface which is not a plane, and consequently falls upon it at different angles of incidence, we get from (15) for each differential element dp, taken with reference to a plane S normal to the direction of the wind,

$$(22) \qquad dp=\frac{0.002698\, s^2}{1+.004\, \tau}\frac{P}{P_0}\cos^2 i dS=\frac{0.002698\, s^2}{1+.004\, \tau}\frac{P}{P_0}\cos^3 i d\sigma$$

$d\sigma$ being a differential element of the surface of the body upon which the wind falls with an angle of incidence i, and which varies with σ.

In the case of a cylinder with its axis normal to the direction of the wind, we have

$$d\sigma=lrdi$$

in which

r=the radius of the cylinder,
l=its length.

Substituting this in the last member of (22), and reducing by the relation of

$$\cos^3 di=\cos^2 id \sin i=(1-\sin^2 i)\, d \sin i$$

and integrating for the whole surface of the cylinder exposed to the wind we get

$$(23) \qquad p=\frac{0.002098\ s^2}{1+.004\tau}\cdot\frac{P}{P_0}\cdot\frac{4}{3}rl$$

This expresses the force of the wind against the cylinder. Since $2rl$ is the greatest section of the cylinder normal to the direction of the wind, by comparing this with (21) it is seen that the force of the wind blowing against the cylinder is two-thirds of that in the case of a surface $S=2rl$ exposed normally to the direction of the wind.

In the case of a sphere, regarding $d\sigma$ as a function of i, we have

$$d\sigma=2r^2\pi \sin i\ di$$

in which

$r=$ the radius of the sphere.

Substituting this value of $d\sigma$ in the last member of (22), and reducing by the relation of

$$\mathrm{Cos}\ ^3i\ \sin i\ di=\cos\ ^2i\ \sin i\ d \sin i=(1—\sin^2)\ \sin i\ di$$

and integrating from $i=0$ to $i=\frac{1}{2}\pi$, that is, for the whole surface of the sphere exposed to the wind, we get

$$(24) \qquad p=\frac{0.002698}{1+0.004\tau}\,\frac{P}{P_0}\cdot\frac{1}{2}\,r^2\pi=\frac{0.001349\,s^2}{1+.004\tau}\cdot\frac{P}{P_0}r^2\pi$$

for the expression of the whole force upon the globe in the direction of the wind. Since $r^2\pi$ is the area of a great circle of the sphere, it is seen by comparing this expression with (21), that the force is only half as much as if the wind blew normally against a surface S equal $r^2\pi$, a great circle of the globe. Loomis[28] obtained from two sets of experiments made at the request of Newton in St. Paul's Cathedral at London, with several hollow glass globes and with several bladders formed into spheres, and all about 5 inches in diameter, a resistance of 0.2856 ounces troy, equal 0.019584 pounds avoirdupois, with a velocity of 15 feet per second, equal to 10.227 miles per hour. With these data in (24), which is adapted to feet, pounds avoirdupois, and miles per hour, putting $P=P_0$ and $\tau=0$, we get p, the force of wind with a velocity of 15 feet per second, and consequently the resistance to a falling body, equal to .019242 pounds avoirdupois, for the theoretical value, which differs but little from 0.019584 pounds, obtained from the experiments. There is some uncertainty, however, with regard to the exact diameters of Newton's spheres.

But in these experiments no account seems to have been taken of the temperature or of the barometric pressure P. The experiments, however, were most probably made in the summer season, and if so, in comparing the theoretical result with that from experiment, we should perhaps put τ in (24) equal to 20° (68° F.). The experiments were made so near sea-level that we can perhaps put $P=P_0$ without sensible error. With these values of τ and of P (24) would give for the theoretical value $p=.017817$, and hence about one-eleventh less than the experimental value. But this was to be expected, since some allowances in such cases have to be made for friction, which is not taken into account in the formula. In measuring forces by pounds, the force of a pound under standard gravity must be understood.

233. The force due to gravity of a spherical body, expressed in pounds avoirdupois, is $62.431\, n\rho.\frac{4}{3}\, r^3\pi$ in which n has its value of (8) § 9, and ρ is its density. Putting this equal to the force of the wind expressed by (24), which is a condition which must be satisfied in order that the body may be supported by an ascending current, we get

$$s^2 = 61660\, \rho r\, (1+.004\tau)\, \frac{P_0}{P} n \tag{25}$$

from which to determine the velocity s of the ascending current which will support a body of given radius r and density ρ, the density of the air, or τ and P, being known. It is seen that the velocity is directly as the square root of the radius and density of the body, and inversely as the square root of the density of the air. Hence it is much greater for high altitudes. It is also as the square root of n and hence a little less in equatorial than in polar regions of the earth.

Putting r equal 1 foot and $\rho=1$, we get for air at sea-level and with $\tau=0$, $s=248$ miles for the velocity per hour required to support a body two feet in diameter and of the density of water. Putting also r equal 1 inch (reduced to feet) and $\rho=0.865$, the density of ice, and $\tau=20°$, we get for the lower strata of the atmosphere, where P may be put equal P_0, $s=69.3$ miles per hour, equal to 101.6 feet per second, for the velocity of the ascending current required to support a globe of ice two inches in diameter. The velocity obtained by Loomis[28], with the numerical coefficient of the formula determined from experiment, is 98 feet per second. This velocity, therefore, would suffice to keep supported in the air almost the largest hail-stones, where the air has the density near the earth's surface, but at great altitudes the velocity required would be much greater.

According to (25) the ascending velocities required to support raindrops and hail-stones of different diameters at an altitude where $P=630^{\text{mm}}$, which is an altitude of about 1 mile, are as given in the following table, for $\tau=0$:

Table.

Rain.		Hail.	
Diameters in inches.	Velocities in miles per hour.	Diameters in inches.	Velocities in miles per hour.
0.01	5.6	0.5	36.6
0.02	7.9	0.6	40.0
0.05	12.4	0.8	46.0
0.1	17.6	1.0	51.4
0.2	24.8	2.0	72.7
0.4	35.2	3.0	89.9

It is seen from these results that a very gently ascending current of air keeps fine rain-drops from falling, and that these are kept up until they unite and form drops sufficiently large to fall down. In falling through the lower strata they overtake and unite with those of smaller diameters falling more slowly, and meet those of still smaller diameters being carried upward, until they arrive where the air is so rare, or the velocity of ascending currents so small, or they become so large by uniting with others, that the force is not sufficient to carry them further up. Hence the further they fall in the cloud region the larger they become. If only large drops fall to the earth's surface it indicates a strong ascending current. It is seen that a velocity of 25 miles per hour prevents drops of 0.2 inches in diameter from falling.

Examples.

1. What is the force of the wind on each square meter of normal surface at sea-level, temperature of freezing, when it has a velocity of 20 meters per second? (20.)

2. What, in the same example, at an altitude of 2,000 meters and temperature of 15°? (20.)

3. What is the force of the wind on each square foot of normal surface at sea-level, and temperature 75° F, with a velocity of 40 miles per hour? (21.)

4. What, with the same velocity, at the height at which the barometric pressure is 17 inches and temperature freezing? (21.)

5. What is the force of the wind, with a velocity of 70 miles per hour, against a vertical cylinder, 40 feet high and 10 feet in diameter, at sea-level and freezing temperature? (23.)

6. What is the force of the wind with a velocity of 100 miles per hour and temperature of 50° F., against a globe 6 feet in diameter, near sea-level? (24.)

7. What velocity of ascending current of air of barometric pressure 29 inches and temperature 60° F. will keep suspended in it a globe of 4 feet in diameter and density 1.5 on the parallel of 45°? (25.)

8. What is the diameter of a hail-stone of density 0.865, which is kept suspended in the air with an ascending velocity of 80 miles per hour, at an altitude of 2 miles and temperature of freezing? (25.)

Hail-storms.

234. If in the expression of h_1, (16), we put for the first term the height of the stratum of freezing instead of that of the temperature of the dew-point, then h_1 will represent, at any distance ρ, the height of this stratum, or R in (18) will represent the radius of the spout of air below freezing temperature at any altitude h_1. We shall, therefore, have in this case

$$(26) \qquad h_1 = H - \frac{v_0^2}{2g}\frac{\rho_0^2}{\rho^2}$$

in which H is the value of h_1 at a great distance from the center. In this case the part of H belonging to the lower strata of temperature from τ' to τ_1 is determined by (12), but from τ_1 to freezing it must be determined by Table XIII in the manner explained below.

If we assume $\tau'_0=30^\circ$, $\tau_1=27^\circ$, $v_0=10^{\mathrm{m}}$ at the distance $\rho_0=1{,}000^{\mathrm{m}}$., we then have by means of Table XIII for the mean value of the rate of decrease of temperature with increase of altitude in this case 0.0042° approximately, between the plane of incipient condensation and that of freezing. Hence we get in this example

$$H=102.1\times 3+\frac{27}{.0042}=6734^{\mathrm{m}}$$

With this value of H (26) gives the values of h_1 for each assumed value of ρ, or (18) gives the values of R for each assumed value of h_1. With the co-ordinates computed either way graphically represented, we get the upper hyperbolic curve of Fig. 6, representing a vertical section of the thin stratum of uniform pressure and freezing temperature. But if instead of this value of H we use only the first term as given by (12), we get in the same manner the vertical section of the stratum of incipient condensation, as represented in the lower curved lines in Fig. 6. The nearly saturated air of the earth's surface is drawn in and ascends on all sides until it arrives at the stratum of incipient condensation represented by the lower curved line, where cloud formation commences. In its further ascent up through the cloud region the aqueous vapor is gradually condensed and the temperature reduced at the different altitudes and temperatures, according to the rates given in Table XIII until it arrives up to the stratum of incipient freezing represented by the upper curved lines in the figure. As soon as the air, with the amount of vapor yet uncondensed, enters anywhere above or within this curve, the vapor left is frozen into snow. This snow, however, never falls to the earth in the summer season, but melts soon after falling below the plane of incipient freezing.

235. In the ascending currents of a tornado, we have seen, the rain-drops arising from condensation down in the cloud-region are kept suspended in the air until they become large enough, in the manner explained in § 223, to fall to the earth. As the vapor in the lower part of the cloud-region is condensed and small drops are formed, these continue to become larger as they ascend by coming in contact with other smaller drops formed still lower down and carried up faster, until by the time they enter within the region of freezing temperature they may be very large drops. These are then frozen solid, and after having been carried up and outwards above to a distance from the center where the ascending currents are not strong enough, by (25), to keep them up, they slowly descend, and receiving additions of ice from the rain as it falls as long as its temperature remains below zero, it finally falls to the earth, a *solid hail-stone*.

But the origin of the hail-stone is often not a rain-drop, but a bunch of snow formed up in the snow-region and moistened by the rain-drops carried up into the snow-region, before there has been time for them to become frozen into hail. This moist snow is kept up there until it freezes and then it still continues to increase from contact with the rain-drops carried up which continually wet its surface and leave a coating of moisture which is at once frozen. When it becomes large, and is carried out where the ascending currents are too weak to keep it sus-

$t_0 = 30°$, $t_1 = 27°$, $v_0 = 10^{m}$:

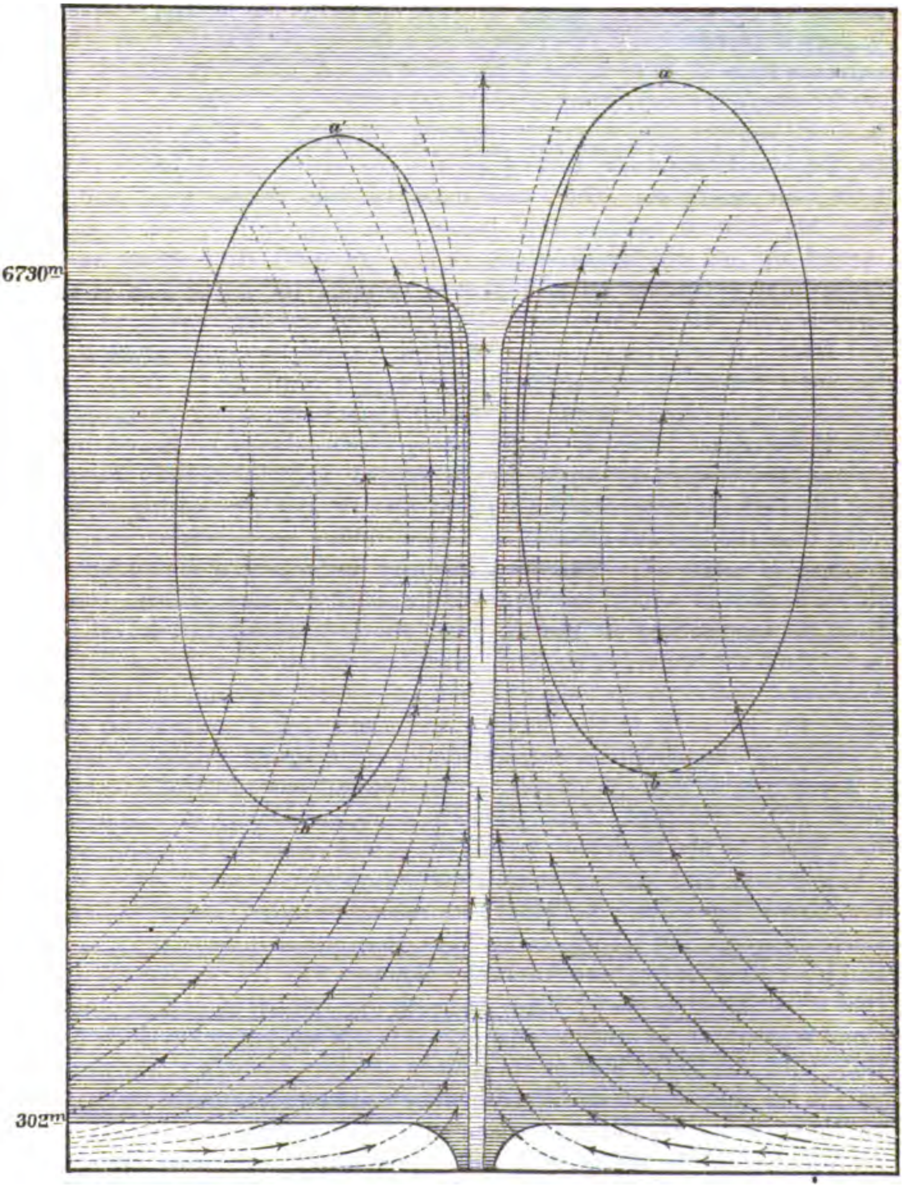


FIG. 6.

pended it falls to the earth a hail-stone, with a *kernel of frozen snow in its center.* In falling to the earth down through the cloud-region with temperature in the lower part much above freezing, of course its original size is somewhat decreased. It often happens that in falling very gently where the ascending currents are not quite, but nearly, sufficient to keep it suspended, it descends so near the center that the indrawing currents from all sides in the lower cloud-region draw it in towards the center where the ascending currents are sufficient to carry it up again into the snow-region, where it receives a coating of snow, moistened by the small rain-drops carried up into the snow-regions before they have time to freeze. The whole mass becomes frozen solid, and, it may be, reduced considerably below zero temperature, before it is carried up and out again where it can gradually drop down again toward the earth, and in falling, even through the lower part of the snow-region, where there is little snow, but mostly rain-drops not yet frozen, it receives a coating of solid ice, for its temperature having been reduced considerably below zero, it continues to freeze the rain coming in contact with it for a long distance in its passage through the cloud-region. But in gently falling down it may be drawn in a second time toward the center and be carried up by the rapidly ascending currents into the snow-region and receive another coating of wet snow over the last one of solid ice, and in falling again receive another coating of solid ice. This process may be repeated a number of times, in each of which the hail-stone describes a kind of orbit somewhat as represented in Fig. 6, until finally it is carried out above so far from the center, or the strength of the tornado becomes so much weakened, that it is no longer carried in toward, and up, in the central part, but falls to the earth, a hail-stone with a *snow-kernel and a number of alternating concentric strata of solid ice and frozen wet snow.*

A remarkable example of this last kind of hail was observed in a hail-storm at Northampton, Mass., June 20, 1870, an account of which has been given by Mr. Houry.[29] In this storm hail-stones fell weighing over a half pound, and most of them were formed of concentric layers like the coats of an onion. Mr. Houry states that in one of them he counted thirteen layers, indicating, as he says, that it had passed through as many strata of snowy and vaporous clouds. The true explanation is that it oscillated as many times between the rain-cloud and the snow-cloud regions, or, in other words, that it performed six or seven revolutions with the lower part of its orbit in the rain-cloud, and the upper part in the snow-cloud, as described above and as represented in Fig. 6. In connection with the vertical and radial components of motion, as represented in the figure, of course there is also the rapid gyratory motion belonging to the interior parts of cyclones and tornadoes.

236. From what precedes, a hail-storm differs from any ordinary tornado, only in its having ascending currents sufficiently strong to carry large rain-drops up very high into the region where the temperature is

sufficiently low to freeze them. Although this region is much higher in summer than in winter, yet we have hail mostly in the former season, since in the latter season we rarely have the conditions which give rise to a tornado of sufficient violence to carry up rain-drops very high, and besides the stratum of incipient freezing then is so low that there is little or no rain-cloud region in which large rain-drops can be formed and in which, after having been frozen above, they are increased in falling through. Winter hail, therefore, consists of only very fine particles of frozen water.

As in a cyclone and a tornado, large rain-drops are an indication of strong ascending currents, so hail-stones indicate a still much stronger current of that kind.

Modifying effects of friction.

237. So far we have taken little account of friction, especially of that between the air and the earth's surface. In all cases where there is any disturbing power, however small, arising from differences of temperature it has been necessary to suppose a small amount of friction, just sufficient to prevent a continual acceleration of motion and to reduce the gyratory velocities to the same at all altitudes, whatever may have been the initial velocities. The great power in the center of a tornado does not depend much upon any forces arising from differences of temperature, but upon that which is already contained in it by virtue of its initial gyratory state, which gives a value to the constant $\rho_0 v_0$ contained in our expressions. The small forces arising from differences of temperature between the central and exterior part in the tornado, as well as in a cyclone, except in the initial motions, is spent in overcoming friction, but still by means of this the force which is already in the tornado is concentrated, in accordance with (5), into a comparatively small area at the center at the expense of that of the surrounding parts. If we put $\rho_0 v_0 = 0$, that is, suppose there is no initial gyratory motion, then the only power we get is from the action of the force arising from differences of temperature, which causes a continually accelerated radial velocity, toward the center below and from it above, until the friction becomes sufficient to counteract the force, after which the velocity becomes uniform. In this, however, there is no gyratory velocity developed, and no concentration of power in the central part, at the expense of the surrounding parts.

238. The effect of friction in tornadoes is much less than in cyclones. A cyclone of considerable extent may be regarded as a disk, with a diameter many times greater than its depth or thickness, and hence the gyrations of the air are very much retarded by friction between it and the earth's surface; but a tornado is rather a pillar of gyrating air with a small base in comparison with its altitude, and hence the retardation of the gyrations by friction, except at the base at the earth's surface, is comparatively very small, and they are very nearly in ac-

cordance with the principle of the preservation of areas, as expressed in (5), and consequently must be very great, but of course some less than in the case of no friction. All the preceding results, therefore, obtained upon the hypothesis of no friction, are somewhat modified in the real cases of nature, but not nearly so much as in the case of cyclones.

According to the results which we have obtained upon this hypothesis, the whole column of gyratory air in a tornado is somewhat like a tall flue with rarefied air within it, but with the draft cut off below, for the centrifugal force arising from the rapid gyrations act as a barrier to prevent the flow of air from all sides into the interior, not only above, but likewise down to the earth's surface. In this case there would be only a very gently ascending current in the interior arising from difference of temperature, the velocity of which at most would not equal that given by (11), since this is obtained by integration of (10), without taking into account the condition of continuity, which causes a heaping up of the atmosphere over the top of the column, which retards the upward velocity and causes a deflection toward the top into lateral horizontal ones.

But if we now suppose the gyrations at the bottom to be considerably retarded by the friction of the earth's surface and the centrifugal force consequently diminished accordingly, it is readily seen that this must allow a rush of air in below, which must escape up in the interior. It is the same as supplying a draft to the flue, which before had been almost completely cut off. It is in this way we get the powerful ascending currents in the interior of tornadoes necessary to account for the great supporting power which they are known often to possess.

It is seen from § 223 that the greater $-D_hP$, that is, the increase of the rate with which the pressure decreases with increase of altitude, the greater is the rate of vertical acceleration of velocity, and consequently the greater the increase of velocity in ascending through any vertical distance corresponding to a given decrease of pressure. It is seen from an inspection of Fig. 1 that in the central part of a tornado the pressure is much diminished, and the strata of equal pressure being nearly vertical, the vertical change in pressure is also small. A great part of the outside normal pressure is resisted by the centrifugal force of the gyrations. If this force were all cut off below near the earth's surface by an entire absence of gyratory velocity, the pressure then at the center would be equal the outside pressure, except so much of it as would be used in overcoming the friction of a rapid current, and then there would be an enormously rapid ascent, and even where the gyratory velocity is only considerably diminished the ascending velocity must be very great.

If we suppose the barometric depression in the middle of a tornado to be only 30^{mm} without any gyratory motion at the earth's surface, so that there would be no force to resist the outward pressure, then by means of (13′), putting $P_0'=760^{mm}$ and $\tau=30^{\circ}$, we get the value of s, or x in this case, equal 84^{m} for the velocity per second of the ascending current. This is a case which could occur if the gyrations were over a very deep and narrow valley. The outside pressure would then come in unobstructed, except by friction, from both sides along the valley.

We have in this case no increase of power, but, upon the whole, a diminution of it, and the increase of the ascending velocity in the interior is at the expense of the gyratory velocity which it had before. The gyratory velocity being diminished below, the motion toward the center below becomes more nearly radial and much more ascensional. It is the same in the case of cyclones in which, we have seen, the effect of friction near the earth's surface is to cause a much greater inclining of the directions of motion toward the center, and to allow the air, which supplies the gently ascending current in the interior part, to flow in mostly very near the earth's surface.

239. The effect of friction being to diminish power, all the effects which we have obtained upon the hypothesis of no friction must generally be a little, and often very much, greater than the true ones and observed ones where there are observations. The gyratory velocities do not follow strictly the law of (5), even far above the earth's surface, and near it they may deviate very much from it, especially on land. The velocities being diminished by the effect of friction, it is readily seen from (13) or (13′) that the computed diminution of pressure corresponding to any assumed value of the constant $\rho_0 v_0$ is too great.

According to (16) and (18) every tornado must have a water-spout in its central part, larger or smaller according to the amount of initial gyration which gives value to the constant $\rho_0 v_0$, and the amount of aqueous vapor in the air, upon which the value of $(\tau'-\tau_1)$ depends. With a small gyratory velocity and little vapor it might be very slender and threadlike extending down to the earth's surface, but in no case could the value of R in (18) vanish. It is a co-ordinate of the hyperbola, approximating but never touching its asymtote. But the effect of friction is to retard the very rapid gyrations near the center; and since the effect is comparatively very great there, the velocities are not so great, that the pressure, and consequently the temperature, are sufficiently diminished to cause condensation even in the center, and thus to give rise to a water-spout.

When the spout first appears it is seen as a funnel-shaped cloud suspended from the lower surface of the undisturbed stratum of clouds. As the gyrations become more rapid, it descends lower, until finally it extends to the earth's surface. This descent may be very rapid, since only a small change often in the gyratory velocities is required to diminish the pressure and temperature in the middle sufficiently to cause condensation in a small column. After continuing some time until the violence of the tornado begins to diminish, and the force arising from the gyrations is no longer able to keep the pressure and temperature in the center low enough to cause condensation, the lower part of the spout vanishes, and with only a very little more decrease of power it may be suddenly drawn apparently up into the clouds above, or at least so much as to leave the appearance only of the funnel-shaped cloud. Sometimes, as it is carried along in the general currents of the air, the gyrations are suddenly increased or diminished, from variations in the amount of friction at the earth's surface, from air being drawn in which has a greater gyratory motion with reference to its center, or, it may be, from various other causes. When this is the case the spout is seen to apparently drop down and rise up suddenly, and this may be repeated several times. It must not be supposed, however, that there is a real dropping down or rising up of the cloud, but simply a sudden bringing about and vanishing of the conditions all the way down which give rise to condensation and cloud-formation. The spout is sometimes seen to lag behind, and not quite reach to the earth's surface. This arises from the general currents above moving faster than those at the earth's surface, so that in order to maintain the spout all the way down it is necessary to continually bring into the gyrations new portions of air. The tornado often does not have sufficient power to overcome the inertia and friction of the gyrations of these new portions of air, and hence the gyrations then are not sufficient to bring the spout down to the earth. In fact it can never happen in nature that we have all the regular conditions assumed in the preceding theoretical treatment, or even a near approximation to them, and hence we can never expect to have a regular theoretical spout such as represented in the preceding figures.

240. In the force of the wind and the supporting power of ascending currents the neglect of friction in the theory causes it to give results *too small* instead of too great. The effective force of the wind against a plate or barrier of any kind does not depend simply upon the inertia of the air, but upon the dragging effect of the air through friction upon the column of air in the front and rear, partially brought to rest by the barrier. This increases the pressure a little in front of the barrier and diminishes it in the rear. This latter is seen in the effect of cowls placed upon the tops of the flues of chimneys. The air is dragged away and the pressure diminished, so that the draft of the flue is increased. Hagen's empirical formula, determined from very accurate experiments made only a few years ago, gives for the pressure in grams on small plates,

$$p=(0.00707+0.0001125\,u)\,F\,v^2 \tag{27}$$

in which u is the periphery and F the surface of the plate in decimeters, and v is the velocity per second in decimeters. The barometric pressure in the experiments was 758^{mm} and the temperature 15°.

This formula, expressed with the measures and notation of (21), so as to give the temperature and pressure corrections, is

$$(28) \qquad p=\frac{(0.003064+.0001479\,u)\,s^2}{1+.004\,\tau}\cdot\frac{P}{P_0}S$$

Distinguishing p with an accent in this case, we get by comparing this with (21)

$$(29) \qquad p'-p=\frac{(.000366+.0001479\,u)\,s^2}{1+.004\,\tau}\cdot\frac{P}{P_0}S$$

as the effect of friction upon the effective force of the wind in moving the plate. About one-half of this, perhaps, is due to increased pressure upon the one side of the plate, and the other to a relieving it of pressure on the opposite side.

It is seen that this has a term depending upon the periphery, and hence it increases in a greater ratio than the extent of surface, and for very large plates it would be very great. The experiments were made with small plates from two to six inches square, and most probably the formula cannot be extended to larger areas.

The same correction is applicable to the supporting power of ascending currents on a normal surface, but is no doubt different in the case of a sphere supported in the air. We have seen, however, that in the case of hollow globes falling through the air the theoretical resistance was one-eleventh less than the experimental one, which does not differ much from the effect of friction in the preceding case, (21) being less than (28), where u is small, by about one-eighth part of the latter. This makes the effect of friction in the case of a sphere less than in that of a square plate of the area of a great circle of the sphere, which is what we would reasonably expect.

Precipitation and cloud-bursts.

241. On account of the very great velocity of the ascending currents of saturated air in a tornado, § 238, the aqueous vapor of which is nearly all condensed, the amount of precipitation, rain, hail, or snow, must often be enormous. Let

e=the elastic force of aqueous vapor at the base of the cloud,
d=the depth of rain in a unit of time resulting from the condensation in the ascending current, supposing that the current extends so high that the vapor is all condensed.
u=the ascending velocity per second at base of the cloud.

Then the weight of vapor in a column of homogeneous air measured by the height of the mercurial column of the barometer is $0.623e$. Multiplying this by 13.6, the density of mercury, and dividing by 7992, the height of a homogeneous atmosphere, we get $.00106\,e$ for the depth of water equal to a column of air of one meter in altitude. Hence if the vapor of the saturated air at the base of the cloud is all condensed in its ascent, the fall of rain per second is

$$d = .00106\,eu$$

If in the example of Fig. 3 we suppose the ascending velocity at the base of the cloud to be 60 meters per second, the vapor tension of saturated air at the temperature $\tau = 25^\circ$, by Table X, being 0.02355^{m} we get

$$d = .00106 \times 0.02355 \times 60 = .0015^{\mathrm{m}}$$

for the depth of rain per second or $.09^{\mathrm{m}}$ (3.6 inches) per minute for the fall of rain, if it were all to fall directly back. From the reasoning of § 238, and from what we know of the supporting power of ascending currents in tornadoes, the assumed ascending velocity of 60^{m} per second is no unusual one, and therefore this result gives us some idea of the immense quantity of rain which may fall in a tornado in a short time. Of course the vapor is never carried so high that all of it is condensed, and being carried out above from the center, the area of rainfall is generally much greater than a section of the column at the base of the cloud where the velocity of ascent is so great. But still it must be considered that the currents above the base of the cloud are bringing in vapor from all sides into the central column of rapidly ascending air, so that the vapor which enters the base of the cloud may not be half of the whole quantity which comes into the central ascending current and is condensed.

Where the ascending velocity is very great the rain does not fall in the central part at all, but is supported until it is carried out above, where the ascending currents are not sufficient to keep it from falling, and it falls around about the more active central part of the tornado. With an ascending velocity of 60^{m} per second (134 miles per hour) the rain would all be carried up to an altitude where it would be carried out from the vortex before falling. According to (25) this velocity at an altitude where the pressure of the air is diminished one-half, and where consequently $P_0 : P = 2$ in the formula, would support a rain-drop 3.84 inches in diameter, which would be increased a little from the effect of friction, §240. With such an ascending velocity, therefore, no rain could fall back in the central part, but there would be an immense rainfall around in its vicinity, especially if the tornado in its irregular progressive motion should remain stationary, or nearly so, for several minutes at any place.

242. If the velocity of the ascending current is not so great that the rain is all carried up to where the currents are outward from the vortex, and where consequently the water is dispersed, and yet great enough to prevent its falling back, then in the whole of the lower part of the cloud in the central part of the tornado, even up perhaps to the altitude of 3 miles or more, there may be a great accumulation of rain, prevented by the ascending currents from falling, and also by the centrip-

etal horizontal component of the current from being carried out and dispersed. Of course the sustaining of this water in the cloud uses up the energy of the tornado and hastens its breaking up. Suppose the gyratory motion and all the circumstances of the tornado were such as to give a depression of 30^{mm} as assumed in §238, but that the gyrations at the earth's surface were so diminished as to reduce this to 15^{mm}, then there would still be an effective force of 15^{mm} for giving rise to an ascending velocity in the interior. Now if rain were so collected in the body of the cloud as to reduce this effective force from 15^{mm} to 3^{mm} this would require a depth of water equal to $12^{mm} \times 13.6 = 163^{mm}$ (about 6.3 inches) and the 3^{mm} remaining, by (13′) would still give an ascending velocity of 27^{m} per second, or $27 \times 2.237 = 60$ miles per hour. This by the table of section 233 would still support rain-drops with a diameter equal to 1.6 inches at the height of 1 mile, but less at greater altitudes. Hence a depth of rain even greater than 6.3 inches would be required to so diminish the effective force in producing ascending velocities that these should not be sufficient to prevent drops of ordinary sizes from falling to the earth. If we now suppose the energy of the tornado and the velocity of ascending currents to be somewhat rapidly diminished either by friction or by the accumulation of water in the cloud, this accumulation may nearly all fall to the earth in a very short time and give rise to a *cloud-burst.*

This is especially liable to occur in mountainous regions; for if we suppose that a tornado thus heavily charged with rain is moving toward the side of a mountain, its coming in contact with it would interfere very much with the gyrations and energy of the cyclone and tend to break up the whole system almost at once and let the whole accumulation of water drop suddenly down. Hence cloud-bursts most frequently occur on mountain sides.

243. The water in cloud-bursts does not generally fall as rain, but is *poured down.* Long before the ascending currents are so reduced as to allow the water to fall in drops it seems to collect together at certain places and force its way downward through the ascending current in a stream. This it would naturally do, since we cannot suppose that the water is ever evenly distributed over any given place, or that the velocities of the ascending currents are the same at all places in the same vicinity. A considerable body of water having been collected at certain points, it is then enabled to force its way down, and it draws into its train much more from all sides on its way, so that it may become a continuous *stream* of water, kept up for several minutes. Of course, having collected in large masses and once made an opening for itself at one or more places, its velocity is gradually accelerated, since the ascending currents then are not able to support them, so that on reaching the earth the velocity may become immense and the stream strike with great force. Each one of these descending streams may make a great hole or basin in the ground; and on a steep mountain side, if the stream continues

only for a short time, it may give rise to a mountain slide, or at least to a great ravine, and carry rocks and trees with it down the mountain side.

Immediately after the great tornado at Hollidaysburg, Pa., on the 19th of June, 1838, Espy[5] visited the vicinity and examined the sides of the ridges and mountains. He found a great many holes eight or ten meters in diameter and one or two in depth, according to the nature of the soil and depth to the rock, and the sides often cut almost perpendicularly down on the upper side, but entirely washed out below, so as to form the commencement of a ravine. Sometimes the current descending through the atmosphere seemed to strike the earth with so great force that it made a great hole or basin and then rebounded so as not to strike the earth again on the mountain side for a considerable distance. A considerable number of these holes were often found close together in the same vicinity, indicating that the water was poured down at the same time through several openings made in the ascending air currents beneath by the concentration of large bodies of water at these places.

With regard to a ridge a half mile west of Hollidaysburg, he says:

> On examining the northern side of this ridge, large masses of gravel and rocks and trees and earth, to the number of twenty-two, were found lying at the base on the plain below, having been washed down from the side of the ridge by running water. The places from which these masses started could easily be seen from the base, being only about 30 yards up the side. On going to the head of these washes, they were found to be nearly round basins from about 1 to 6 feet deep, without any drains leading into them from above. The old leaves of last year's growth, and other light materials, were lying undisturbed above, within an inch of the rim of these basins, which were generally cut down nearly perpendicularly on the upper side, and washed out clean on the lower. The greater part of these basins were nearly of the same diameter, about 20 feet, and the trees that stood in their places were all washed out. Those below the basin were generally standing, and showed by the leaves and grass drifted on their upper side how high the water was in running down the side of the ridge; on some it was as high as 3 feet. It probably, however, dashed up on the trees above its general level.

In an account of a remarkable storm which occurred at Catskill, July 26, 1819, it is stated that "the rain at times descended in very large drops, and at times in streams and sheets."

It is evident from these and many other similar paragraphs which might be cited, the water in such cases is poured down in streams, and does not fall as rain, especially since the lightest materials close to the margins of these basins on the upper sides, referred to above, were not washed away; and the reason of this, evidently, is that the accumulated water in the cloud is poured down in streams, while the ascending currents are, as yet, too strong to allow the largest drops to fall as rain.

244. It is seen from a reference to the table of § 233, that a raindrop 0.2 inches in diameter at an altitude of 1 mile cannot fall through an ascending current of air with a velocity of 25 miles per hour, and those with still smaller diameters are carried up to where the velocity and

density are such that they are supported by them, and only begin to fall after they become so large, or the velocity of the ascending current becomes so much diminished, that they are no longer supported. Hence, with an ascending velocity of 25 miles per hour, no rain in drops of ordinary size can fall directly back to the earth, but in the case of a tornado it can be carried out above, and fall at a distance from the center, provided that the drops are so small that they can be carried up far above the position where the ascending currents turn outward from the center. From what precedes, a velocity of 25 miles per hour would carry most, if not all, the rain in the central part of the tornado up to that altitude, which was not formed into drops with diameters greater than 0.1 inch. Those with larger diameters would have to remain in the cloud in the central part of the tornado until they became much larger, or in the case of very rapid ascending currents until they collected into large masses and forced themselves down in streams of water.

245. All that has been stated with regard to immense rainfalls and cloud-bursts is applicable, with a few modifications, to the sudden falls of immense quantities of hail. In this case, however, the circumstances must be such that the out-turning ascending currents are at a great altitude and the ascending currents are so great that instead of the accumulation being down very far below the region of freezing, it must be much higher up but not necessarily up in the region of freezing. For the hail-stones, formed in the manner described, § 235, may fall directly back from above or be carried out above and come around and be carried up into this place where the ascending currents are too strong to allow them to fall and not strong enough to carry them up where the currents would carry them out above where they can fall, and so they remain there a considerable time without much loss from melting, and if from any cause the energy of the tornado becomes exhausted suddenly, the accumulation of hail all falls as rain does in a cloud-burst, in a very short time, leaving a great depth of hail over a small district of country.

Where the bed of hail is below the region of freezing of course the hail-stones cannot increase in size, but must rather diminish. In some extraordinary cases, however, the ascending currents may be so strong, and the altitude where they turn out from the center so great, that the collection of hail is up in the freezing region, and then by continually freezing the rain carried up from below and coming in contact with them, they grow until they become so large, or until the velocity of the current is so much diminished, that they can be no longer supported. This may not be until they become very large. Accordingly hail-stones of enormous size are sometimes observed.

At Olmutz, on the 31st of May, 1868, there was a hail-storm in which the larger stones were mostly of an irregular and oval shape with uneven surface, the longest diameter being from 14 to 22 lines. Those from 18 to 22 lines all weighed over 3 drams, and the greatest, 24 lines in length, weighed 5 drams.[30]

In Iowa, in April, 1880, "hail-stones 12 inches in circumference have been measured in Sac County. In Davis County a flattened disk of hail measured 4½ inches in diameter and was 2 inches thick. In Iowa County a group of ice crystals fell, 2 inches in length, 1¾ inches wide, and one inch thick."[31]

There is a great diversity in the shape of hail-stones. Some are like a disk or very oblate spheroid, and others are oblong. If for any reason the hail-stone becomes the least flattened or oblong, the ascending current which keeps it suspended in the air also keeps its shortest dimension perpendicular to the direction of the current, and hence it increases most on its edges or ends, where the ascending rain water adheres and becomes frozen more readily than at the sides, where it is carried rapidly past by the current.

They often appear to be fragments of broken ice of almost any shape, with the angles more or less rounded off. This can only be accounted for by supposing that very large hail-stones are broken into fragments by clashing together in the vortex of the tornado. In falling a long distance, often slowly through the ascending current, the sharp corners are melted away.

A remarkable example of this kind was related in a letter to General Sabine by Captain Blackiston.[32] It was a shower of ice in a tornado, which occurred January 14, 1860, about 300 miles SSE. from the Cape of Good Hope, and continued about three minutes. It was said to be not hail, but irregularly formed pieces of solid ice of different dimensions up to the size of a half brick. Some weighed from 3½ to 5 ounces after a considerable part had been melted away.

Fair-weather whirlwinds and white squalls.

246. These are simply small tornadoes, which occur during fair weather whenever and wherever the proper conditions are found. These are found when there is an unusually rapid decrease of temperature with increase of altitude, or else a very nearly saturated atmosphere. They are sometimes observed on land, but mostly on lakes and at sea, and may be accompanied by a water-spout or not, according to the violence of the gyratory motion and the amount of aqueous vapor in the atmosphere. At first a small cloud is formed in the clear sky, which gradually increases from the condensation of vapor rapidly carried up in the central part of the gyrating air, which in this case arises just as in the case of any other tornado. The gyrations may sometimes continue to increase in violence and extent, and the cloud to spread until it assumes the dimensions of a tornado such as usually takes place in a cyclone, but it is mostly confined to only a very narrow streak, and its violence, although very great at the center, is not felt at a short distance away. Although the gyrations are violent very near the cen-

ter, yet decreasing in velocity, at least a little above the earth's surface where the friction is small, somewhat according to the law expressed by (5), at a short distance from the center they are entirely harmless, and they may pass very near without doing any injury. If, however, they pass directly over houses on land, or a ship at sea, their destructive effects may be very great and sudden.

When these small tornadoes are accompanied by a water-spout and are first seen at sea at a distance, they can generally be avoided, or if not, preparations can be made to guard against danger; but where they are suddenly formed on the spot and drop down, as it were, from above, they give no forewarning of approaching danger. A remarkable example of this kind was related in the New York Herald of December 10, 1878. The British bark Bel Stuart, Captain Harper, on the evening of November 14, 1878, 160 miles from Cape Sable, "was struck by a white squall in a comparatively smooth sea and clear sky, which swept her decks and created consternation on board. At 6 p. m. of the same day, all hands being on deck after supper, a strange sighing of the wind was observed by the watch, and the sky became suddenly threatening without corresponding indication of the barometer, which showed a rising tendency; Captain Harper and his first officer were on the deck at the time. All hands noticed the peculiar change in sea and sky and were discussing it, when, without a moment's notice the sea forward seemed to swell up to meet the lowering sky and swept the bark across her bows, carrying away her foretopgallant mast, jib, jib-boom, foretopmast-stays, and the maintop-gallant mast, with all their accompanying sails. In a moment, as it seemed, the bark, with all sail set in a fair wind, with a moderate sea, was left a comparative wreck to wallow in the trough of the tremendous seas which had followed the spiral volume of water. Two minutes before the fatal catastrophe, Captain Harper says there was no indication of the water-spout."

It seems from this account that the bark ran into the spout as it was being formed and before it became visible. The sighing of the wind was caused by the rapid gyrations of the air, and the threatening sky, by the incipient condensation of the aqueous vapors carried up by the ascending current. The barometer, before its very near approach, remained unaffected because, as we have seen, the barometric pressure is diminished only at and very near the center, and it was by this that the uprising of the sea was caused which swept across the bark.[10]

247. These small tornadoes are sometimes called *white squalls*, especially where they arise in a dry atmosphere and there is formed, at least at first, a small white cloud only, high up in the atmosphere. In such cases, the air being very dry and consequently the aqueous vapor in it not being condensed until it has ascended very high, the gyrations are often not sufficient to bring the cloud down to the sea in the form of a water-spout. There is, no doubt, a funnel-shaped depression of the cloud in the center, but to an observer nearly under it this is not

observed. Although there is no spout, yet the usual rapid gyrations and ascending current are there, as manifested by the rising and boiling of the sea immediately under the small cloud. This cloud, though usually small and white at first, may eventually extend over a considerable portion of the heavens, and where the atmosphere is not too dry, be followed by a considerable fall of rain.

Peltier says:

> White squalls are very rare, but they are sometimes met with between the tropics, especially near elevated lands. They are generally violent and of short duration. They often take place when the sky is clear and without any atmospheric circumstances giving notice of their approach. The only thing which indicates their proximity is the boiling of the sea, which is very much agitated by the violence of the winds. Many of these squalls, which commence either by a little cloud, or even without any visible cloud, are soon accompanied by violent rains and thick clouds.

On the west coast of Africa these little tornadoes or whirlwinds are called *bull's-eye squalls.* According to Piddington [33], the Portuguese describe such a squall as "first appearing like a bright white spot at or near the zenith, in a perfectly clear sky and fine weather, and which, rapidly descending, brings with it a furious white squall or tornado."

From the preceding descriptions it is evident that these squalls differ but little in their nature from the small tornadoes and water-spouts met with in higher latitudes, such as the one which wrecked the bark Bel Stuart, except that the atmosphere is usually too dry for the formation of a water-spout.

248. Small water-spouts observed on seas and lakes in clear, calm, and hot weather usually arise from a state of unstable equilibrium in the lower strata of the atmosphere. In such cases the whirling of the air and the agitation of the water is first observed below, and afterwards the formation of a cloud above, and finally the complete spout, unless the atmosphere is very dry. Where, however, the air is very moist, near the point of saturation, such spouts extend up to only a small height, and are not accompanied by any rainfall, since they are usually too small and continue too short a time to send up the vapor to so great a height that it can be condensed and collected into drops and fall as rain.

A number of such small spouts are often seen at one time in the same vicinity. When the air is brought into the unstable state it is liable to burst up through the strata above at several places nearly at the same time, and then, if there is any whirling motion, even only very small and imperceptible, as in the case of the water in a basin, there is a concentration of this motion into rapid gyratory motions at the center of each one of these upbursts, which may continue to increase until a water-spout is formed.

During the summer season water-spouts of this sort are said to be very common on the Little Bahama Bank, as many as fifteen having been observed at the same time. The following account of them is given by an officer of H. M. surveying vessel Sparrow-Hawk, employed in the West Indies:[34]

> I have noticed that the first movement which eventually produces a water-spout is a whirlwind on the surface of the water, gradually increasing in velocity of rotation and decreasing in diameter as it travels along before the prevailing wind. The spray is lifted to a height of from five to ten feet, and then gradually melts away, assuming the appearance of hot air, which is visible (still rotating) to a similar height above the spray. A motion amongst the clouds soon becomes apparent, a tongue is protruded, and the spout becomes visible from the top downwards.
>
> On one occasion a portion of a spout appeared for a moment in mid-air above the disturbances on the surface of the water.
>
> Although these appearances are commonly called water-spouts, I have been informed by men who have been caught in them that they contain no water and should be properly called "wind-spouts." The small fore-and-aft-rigged schooners that ply on the bank do not fear them, although a prudent captain would probably shorten sail to one. I have been unable to hear of an accident having occurred through a vessel being caught in a water-spout.
>
> They frequently cross the land but no water falls; they take up any light articles, such as clothes spread out to dry, straw, &c., that happen in their course, but have never been known to carry anything with them to a distance.

249. M. Defranc[35] has given an account of a great many water-spouts of the class which occur mostly in weather which is nearly clear and calm. He says:

> On the 23d of June, 1764, a water-spout was seen on the Seine, which had its base on the river and reached up into the clouds. It was judged to be about three feet in diameter where it touched the river. There were some parts transparent, which allowed the ascension of the water to be seen. It finally broke at about one-third of its height. The lower part fell in rain, the upper part was drawn up into the cloud in a second of time and the phenomenon was followed by hail.

This spout is remarkable for the smallness of its diameter, and seems to have arisen from conditions similar to those of Fig. 5, except that there was not so much difference between the dew-point and air temperature, and consequently it was most probably of not nearly so great a height. The air must have been very nearly calm, except some small local disturbances, giving rise to the gyrations as the air of the lower strata burst up at the central point through the strata above and was drawn in from all sides toward that point. We have seen that in the case of no friction the least amount of initial gyratory motion would bring down to the earth a fine thread-like column of very much rarefied and cooled air in which condensation of the vapor coming into it from all sides would take place and give rise to a very slender water-spout. Being a very tall, slender column, with its base on the river, the friction in this and similar cases is very small, and it approximates to the case of no friction. At the time when the air in the central column is not quite rarefied and cooled sufficiently to condense the vapor, the least increase in gyratory velocity brings it into this condition, and then the spout appears to shoot down from the cloud above to the earth's surface. Just the reverse of this occurs at a certain stage when the gyrations are being gradually reduced by friction. The water, of course, rose to a considerable height in the center from diminished atmospheric pressure, and the rapidly ascending currents carried it up still higher, and when it broke suddenly at about one-third of its height this water fell back, but it was not rain. Its being followed by hail indicates that the ascending currents extended high up into the upper strata of freezing temperature. Again he says:

> On May 17, 1863, Captain Cook saw six water-spouts on Queen Charlotte Sound. In one of them a bird was seen, and in arising was drawn in by force and turned around like a spit. Their first appearance was indicated by a violent agitation and elevation of the water. When the tube was first formed or became visible, its apparent diameter increased. It then diminished and became invisible at its lower extremity.

The fact of the bird's being drawn in and whirled around is important in showing that the air is really drawn in from all sides in a spiral and ascending current exactly in accordance with theory. The violent agitation and heaping up of the water before the spouts appeared show that the gyrations and barometric depressions in the center exist before the spout becomes visible, and that the spout appears only after the diminution of tension and temperature necessary to condense the vapors.

250. Small water-spouts which occur in fair weather have been observed to be *hollow*. This is often indicated by the central part of the column appearing lighter than the surrounding parts. It has also been observed, under very peculiar circumstances, by looking down into the top of them. M. Boué,[36] in the year 1850, observed three water-spouts at the same time on Lake Janina, from the top of a high mountain. The weather was entirely clear, without clouds or wind, but very oppressive and hot. The spouts seemed to rise up from the lake, and he could look down into the top of them and see that they were hollow in the middle.

This hollowness in the middle arises, no doubt, from the effect of the centrifugal force of the very rapid gyratory velocity near the center, in driving the condensed vapor, as soon as formed, out from the center, and also from the fact that the central part is so rarefied and cooled down that but little vapor, even before condensation, can approach near it, since it is mostly condensed before arriving there, just as in its ascent vertically into the region of snow and freezing very little uncondensed vapor is left after having ascended to that altitude.

Where and when tornadoes are most likely to occur.

251. It has been shown that there are two principal conditions upon which tornadoes depend, and that in the absence of either of these they cannot take place. The one is the state of unstable equilibrium of the air, and the other a gyratory motion with reference to any assumed center. It is not necessary that this center shall be stationary, but simply that the motion of the air around it shall be such that when it is drawn in toward this center it shall run into a gyration around it. When we have these two principal conditions, the other, that there shall

be some slight initial disturbance to cause the air to burst up at some point through the strata above, can scarcely ever be wanting. The places and times, then, in which these two principal conditions are found are those in which tornadoes are most likely to occur. Of these two, however, the unstable equilibrium is the most important, since it more rarely occurs than the other, which is scarcely ever so entirely absent as to not give at least some gyratory motion which becomes violent very near the center.

252. With regard to fixed areas on the earth's surface where the unstable state is most readily induced at all seasons of the year, these are found where, in the general motions of the atmosphere, as deflected by continents and mountain ranges, currents of air at the earth's surface which come from a warmer latitude, or at sea from a much warmer continent, are caused to flow under the colder upper strata where the normal motion is nearly eastward, and where consequently the temperature is the normal one, not affected by such motion as takes place in the lower strata. One such place is found in the Mississippi Valley, and especially between the Mississippi and the Rocky Mountain range, where, for reasons given in § 171 the air currents of the lower strata are from lower latitudes, comprising the Gulf of Mexico, curving around first northward and then more easterly under the higher upper strata which pass over the top of the range of the Rocky Mountains and directly or nearly eastward without having their temperatures changed from the normal temperature of the latitude by such deflections. And this is especially the case in the summer season when the interior of the continent is warmed up and the air of the lower strata is drawn from lower latitudes far up into the higher latitudes on the eastern side of the Rocky Mountains, and the isothermal curve there is deflected very far toward the north. From this cause the temperature of the lower strata of this region becomes unusually great relatively to that of the strata above; and if the complete unstable state is not induced from this alone, it is readily brought about by the addition of any small effect from some other cause, as from extremely warm weather in which the earth's surface and the lower air strata become abnormally heated. The great moisture of the air in these southerly winds is also favorable to the induction of the unstable state, since it is more readily brought about in air nearly or quite saturated.

The other condition of tornadoes, that of a relative gyratory motion with regard to any point, is also found to an unusual extent in this region, especially in the winter season. For the southerly current curving around toward the east causes a pressure to the right, giving rise to the permanent barometric gradient of increasing pressure toward the central part of the permanent area of high pressure in the Atlantic Ocean east of Florida, and on account of this there is a counter current between this and the Rocky Mountains flowing down toward Texas, just as in the Atlantic Ocean the Greenland current flows down to Florida between the Gulf Stream and the coast of the United States. There is evidence of such a current in this region in the averages of all seasons, since the resultant of these has a large southerly component, and especially is this seen in the isotherms of the Mississippi Valley extending in a somewhat northeasterly and southwesterly direction, so that the mean temperature of New Mexico and the northern part of Texas is the same as that of places east of the Mississippi from five to ten degrees further north. In the summer season this flow of cold air down toward the Gulf is confined to a comparatively narrow belt close to the mountain range, for then the warm currents from the Gulf are drawn further up toward the northwest. At this season the northern part of Texas has the same mean temperature as Minnesota and the isotherms are nearly north and south in direction, and the contrast between the temperature of the warm southerly winds on the one side and the colder northerly ones on the other side is similar to that of the cold wall between the Gulf Stream and Greenland current in the Atlantic Ocean. This tendency of the air, therefore, to flow in contrary directions gives the second condition of a tornado to a greater degree in this region than in almost any other.

Both of the two principal conditions of a tornado, therefore, are found in a pre-eminent degree in the Mississippi Valley, and especially between the Mississippi and the Rocky Mountains. Hence there is perhaps no part of the world where they prevail more than in this region, and especially in that part of it in the middle latitudes west of the Mississippi River embracing Kansas and Missouri.

Near the Rocky Mountain range, and some distance to the east of it, the conditions are not so favorable, both on account of the dryness and the lower temperature of the air. Hence in this region large destructive tornadoes do not prevail much, though small tornadoes with water-spouts are of frequent occurrence.

253. Another place on the globe where the normal condition of the atmosphere approximates to the unstable state is on the west coast of Africa and extending a considerable distance westward over the ocean. Here the warm trade winds from the northern part of Africa run under the comparatively very cold air of the upper strata moving eastward over the Atlantic, § 164, and thus cause a very rapid decrease of temperature with increase of altitude, very nearly, if not quite, equal to that which produces the unstable state in unsaturated air. But the excessive dryness of the air here, coming from the coast of Africa, is a condition which is not favorable to the formation of large tornadoes accompanied by water-spouts, but very small tornadoes or whirlwinds without spouts, usually called white squalls, are very frequently seen in this region and are here called bull's-eye squalls.

254. Tornadoes pretty generally accompany cyclones. This is because the condition of unstable equilibrium necessary in the formation of a tornado is also required in a cyclone, at least in the upper cloud-region.

The gyratory motion, also, of the air in a cyclone furnishes the second of the two principal conditions of a tornado. The unstable state in a cyclone, however, hardly ever extends down to the earth's surface, so that there are not necessarily visible tornadoes and water-spouts in every cyclone; but there are doubtless many secondary whirls in the cloud-region of a cyclone, the effects of which do not reach down to the earth's surface, and the only visible effect above is the formation of a local cloud a little denser and darker than the cloud generally.

It is now known from Finley's researches[27] and tornado charts that the positions of tornadoes in cyclones have a certain relation to their centers, and that they are mostly found in a southeasterly direction from the center. The distance is very variable and depends upon the magnitude of the cyclone and various other circumstances, for in the same cyclone and at the same time there may be several at very different distances, since the conditions of a tornado, at least on this side, may exist at almost any distance from the center.

This relation of the positions of tornadoes to the centers of the cyclone in which they occur is accounted for upon the general principle already given, that the unstable state of the air is most liable to be induced where surface winds from a lower latitude or any warmer region prevail while the normal temperature of the air above is not affected or at least increased in the same way. In a cyclone, we have seen, the air at the earth's surface is slightly anti-cyclonic on the outer border, but upon the whole mostly cyclonic and very much so near the center. With increase of altitude, however, the anti-cyclonic part increases and the cyclonic diminishes, so that at a great altitude it may become entirely anti-cyclonic. On this account we have the relations between the lower and upper currents as represented in Fig. 2, § 195. It is seen from this figure that in the southeast octant, within the cyclonic part of the surface, the surface currents are from the south, bringing warm and moist air northward under the cold air currents above from the north. This increases the temperature below and decreases it above and gives rise to the large vertical gradient of temperature decreasing with increase of altitude which is necessary to the unstable state. And even on the anti-cyclonic part, where the directions are somewhat the same, a similar effect is produced, since the velocities of the upper currents are much the greater.

From an inspection, however, of the northwest quadrant of the figure it is seen that just the reverse takes place. Here we have the colder surface winds from the north under the warmer upper currents from the south, the tendency of which evidently is to diminish the vertical gradient of temperature decreasing with increase of altitude, and hence if the atmosphere in this quadrant were even in the unstable state the effect of this would be to reduce it to the stable state, in which a tornado cannot be found. Hence in this quadrant no tornado can occur and none are observed.

In the southwest and northeast quadrants the currents and counter currents are nearly east and west, and hence the effect is neutral, neither tending to produce or destroy the unstable state.

On the approach of a cyclone, therefore, from its usual direction of west or southwest, we need have no fears of tornadoes on the north or northwest side of its line of direction, and they are most to be feared when the center of the cyclone is in a direction northwest from us. Of course this is only true in the main, and it is not to be understood that the conditions of a tornado may not be found over a large range of district and tornadoes sometimes be observed even on the bordering parts of the southwest and northeast quadrants.

255. For well-known reasons the unstable state occurs mostly in the cloud-region, and hence the tornadic gyrations usually commence there first, and are afterwards propagated downward to the earth's surface. The unstable state, however, is often produced in the lower unsaturated strata of the atmosphere, even when very dry; and then, if the other condition of a tornado is present, small tornadoes at least, may be formed, even where this unstable state does not extend to the upper strata, and thus tornadoes occur not only in cyclones but elsewhere and in clear weather. In such cases, however, the effects do not reach very high into the atmosphere.

The unstable state in unsaturated air occurs mostly on very dry and sandy soils, with little heat conductivity, when the weather is very warm and the heat rays of the sun are unobstructed by any clouds above. The heat thus accumulates in the surface strata of the soil and the lower strata of the atmosphere, and thus is brought about the unstable state, at least up to a low altitude, even in clear and dry weather.

The same is often found, also, in very calm weather over the surface of seas and lakes. The surface of the water becomes heated, and also the lower strata of the atmosphere, by heat rays passing directly down and by those reflected back until the unstable is brought about.

256. The season of the year in which tornadoes mostly occur is that in which the atmosphere in its normal state for the season approaches most nearly the unstable state. This, we have seen from the table of rates of decrease of temperature with increasing altitude, § 140, seems to be in all parts of the world in the summer season, and in the United States at least in the early part of summer, May or June. Hence, although tornadoes occur at all seasons and in every month of the year, yet, according to Finley's researches,[27] "summer is the season of greatest frequency," and "June is the month in which they occur most frequently." The relative frequency of their occurrence in the United States during the winter, spring, summer, and fall seasons was found to be respectively as the numbers 35, 215, 240, and 86. These numbers indicate a great difference between summer and winter in the frequency of their occurrence, and also, since the numbers for spring and summer do not differ much, that the maximum occurs very early in the summer.

For the very same reason that tornadoes occur mostly during the warmest season of the year and very rarely in the winter season, they should occur during the warmest part of the day and seldom at night. For during the day the surface of the earth and the lower strata become very much warmed up, and at night the reverse takes place from nocturnal cooling, while the temperature at a moderate elevation is subject to only a small diurnal variation. Hence, when the general state of the atmosphere, aside from its diurnal variation, is very nearly that of the unstable state, this state is frequently induced during the warmest part of the day by this diurnal and other abnormal variations, but very rarely at night. Accordingly Finley found that "tornadoes are most frequent in the afternoon between noon and 6 o'clock," and that "the hour during which the greatest number of tornadoes occurred was from 5 to 6 p. m.," and that "the next hour was from 4 to 5 p. m."

The same is true with regard to small tornadoes and water-spouts, as well as those which occur in cyclones. M. Defranc[35] remarks that he "never saw a water-spout before 10 o'clock in the morning nor after 5 o'clock in the evening;" that "they never appear during the night nor during the winter, and that there are always two circumstances attending them. The first is the presence of the sun during, or a little before, the phenomenon. The second is the absence of the wind, or only a very feeble one, except in the space occupied by the water-spout."

This refers mostly to small water-spouts on seas and lakes, depending upon the conditions referred to in a preceding paragraph.

257. Tornadoes occur mostly in the summer season and during the warmest part of the day, not only because the vertical gradient of decreasing temperature is greatest at these times, but also because a smaller gradient is required to induce the unstable state then than during the coldest season of the year and the coldest part of the day. By referring to Table XIII, Appendix, it is seen that with a hot surface temperature of the air during the warmest part of the day in the summer season, say 35°, the unstable state for saturated air is induced with a vertical gradient of decreasing temperature of 0.35° for each 100 meters, while if the temperature were that of freezing, the gradient would have to be 0.63° for each 100 meters, and hence nearly twice as great as in the former case. So great a gradient as the latter is rarely, if ever, found in winter even during the warmest part of the day.

In the case of small fair-weather tornadoes and water-spouts, such as are observed on seas and lakes, unless the atmosphere is very near the point of saturation, the vertical temperature gradient required is very nearly that of dry air, which is never found in the winter season, and only during the hottest part of the day in summer. Hence these are never observed at night even in the summer season.

258. Where the unstable state has been very nearly induced from some other cause, as a very hot surface temperature during a warm, clear day, it may often be consummated by the burning of dry brush

on the earth's surface, or of a cane-brake, or by a large fire of any sort, and thus great whirlwinds, and even showers of rain accompanied with thunder, may be produced. The vertical columns often visible in such cases are composed of dark smoke brought in from all sides towards, and carried up in, the center. These assume all the usual forms of water-spouts, and of course often contain also condensed vapor in the form of a water-spout concealed by the smoke, else rain would not be produced.

Mr. Omsted[37] has given an interesting account of such a phenomenon arising from the burning of a cane-brake on the shores of the Black Warrior in Alabama. Columns of smoke of various forms were witnessed which assumed the usual forms of water-spouts, some extending up to only a moderate height with a funnel shape at the top, while others were very slender columns extending up 300 yards into the clouds of smoke, and all were accompanied with a whirling motion of the air and the smoke. Both Redfield[38] and Espy[5] have given a number of statements made by eye-witnesses of the effects of great fires in producing whirlwinds, rain, and thunder, from which it appears evident that they are at times followed by a considerable amount of rain. And it was proposed by the latter to try the experiment of producing artificial rains in time of drought by burning great quantities of brushwood, or by means of prairie fires in the West when the grass is dry.

Sand-spouts and dust-whirlwinds.

259. In very hot, dry climates, where there is a sandy soil, sand-spouts and dust-whirlwinds are of frequent occurrence. The dry air of such climates, especially over a sandy soil, is often in a state of unstable equilibrium from the accumulation of heat on the earth's surface, for a sandy, dry soil conducts it very slowly down into the earth, and there are then generally all the conditions of a whirlwind and a water-spout, except the vapor in the air to condense, for the condition of an initial whirl of the air can scarcely ever be wanting where there is not a perfect calm. The gyrations of the air and the ascending currents are the same as in a waterspout, but instead of aqueous vapor, sand or dust collected and drawn in from the vicinity is carried up. The inflowing and spiral currents from all sides towards the vortex, up to a considerable height often, keep it near the center in the form of a column or pillar of sand, if the whirlwind is well developed over a small area only with very rapid gyrations and a strong ascending current. The height of the column depends upon the strength of the ascending currents and the altitude at which they are turned outward from the vortex, for, as in cyclones and water-spouts, where there is a flowing of the air in from all sides below, it must flow out again above a certain altitude, depending upon the different circumstances under which the whirlwind takes place. Sand-spouts are frequently observed in Arabia, Persia, and India, and also in Arizona and other places in the western part of the

United States where the climate is very dry. In the hot, dry climate of Australia, situated in the dry zone of the southern hemisphere, § 166, these pillars of sand are said to be often more than a half mile in height.

Where the whirls take place over a considerable area, and do not become concentrated into rapid gyrations near the center, and the ascending currents do not extend up very high, they give rise to dust-whirlwinds, which often overtake caravans and travelers on the deserts, and if they produce no fatal effects they are at least very unpleasant and annoying. These are very frequent in India and the Sahara or Great Desert of Northern Africa, and in fact in all places in the warmer latitudes where there is a dry, sandy soil.

Where the air is nearly calm the sand and a thin stratum of atmosphere in contact with it become heated very much above the ordinary temperature of the air a little above the surface. The inflowing currents of the whirlwind from all sides collect this warm surface stratum into the central part of the whirlwind and cause the whole interior to be of an extraordinarily high temperature. The much-dreaded simoon, stripped of all exaggerations, is most probably simply one of these dust-whirlwinds.

260. Sand-spouts, as well as water-spouts, have been observed to be hollow. Of a whirlwind observed at Schell City, Mo., in the summer of 1879, it was said,[39] "there were no surface winds strong enough to bear dust along the surface of the ground, but the dust carried up in the vortex was collected only at the vortex of the whirl. The dust-column was about two hundred feet high, and perhaps about thirty or forty feet in diameter at the top. The direction of rotation was the same as of storms in the northern hemisphere. Leaving the road the whirl passed out on the prairie, immediately filling the air with hay, which was carried up in somewhat wider spirals, the diameter of the cone thus filled with hay being about 150 feet at the top. It was then observed also that the dust-column was *hollow*. Standing nearly under it, the bottom of the dust-column appeared like an annulus of dust surrounding a circular area of perfectly clear air. The area grew larger as the dust was raised higher, being about fifteen or twenty feet wide when last observed."

The sand-spout and the dust-whirlwind, where well developed and concentrated, is free from sand or dust in the center for the same reason that the water-spout is free from cloud or condensed vapor. The centrifugal force of the gyrations keeps it off at a distance where this force is just equal to that of the indrawing currents which tend to drive it in toward the vortex.

Blasts of wind and oscillations of the wind-vane.

261. The wind is often observed to blow in blasts, with an oscillating vane and unsteady barometer. This arises from the air running into

numerous whirls, or gyrations, while it at the same time has a progressive motion. As in a cyclone, while passing over any place, the wind is first from one direction and then, in the course of a day or two, gradually veers around to another, often to one in nearly a contrary direction, so in the passages of small whirlwinds, the vane in like manner oscillates from one direction to another in a few minutes, the manner and range of oscillation, as in the case of a cyclone, depending upon which side of the vane the center of the whirl passes, and the distance from it. If the center of the whirl passes over the vane, then there is a very sudden oscillation of the vane through a range of 180°, or nearly, especially where the diameter of the whirling air column is small.

As in a cyclone the velocity of the wind is the greatest on the side called the dangerous side, on which the direction of cyclonic motion coincides with that of the general progressive motion, and is comparatively small, or may entirely vanish, on the opposite side, so in one of these small whirls of air the velocity on the one side is very much increased above the average, which is that of the general progressive motion, while on the other it is as much diminished, and there may be almost a calm. Wherever the air is in the unstable state these little whirls are very numerous, and consequently, in their passage over any place, they cause the wind to blow in blasts, and a very frequent oscillation of the vane from side to side occurs, sometimes from right to left and at other times in the contrary way.

These little whirls in the atmosphere are especially liable to occur in connection with cyclones extending over a considerable area, for in these, especially up in the cloud-region, the air is in the unstable state, and on account of the gyrations of the cyclone and the general agitation of the air, the moments of gyration, with regard to any point where there may be a rushing up of the air of the lower strata through those above, can scarcely ever be 0, and hence there are both of the two principal conditions of such little whirlwinds. These may form small secondary cyclones contained within the larger, or tornadoes and water-spouts, or simply local whirlings in the atmosphere of small extent and no great violence, but sufficient to cause intermittences in the steadiness of the velocity of the wind and oscillations in its general direction. Hence there is generally great unsteadiness in the velocity and direction of the wind in a cyclone, and the greatest injuries usually arise from the violence of the wind on the side of these subsidiary whirls where the direction of motion coincides with that of the gyratory motion of the cyclone.

These blasts and oscillations of the vane are generally observed on the clearing-up side of a storm. As the central area of a cyclone is warmer than the surrounding parts, and the upper colder strata in middle and higher latitudes move eastward faster than the lower strata, the effect is to cause a more rapid decrease of temperature with increase of altitude, and hence to induce the unstable state which gives rise to these whirls. As the air is comparatively dry on this side of the storm the small amount of vapor remaining is usually carried up in the central part of the whirl to a considerable altitude before condensation takes place and then it forms a patch of whitish fracto-cumulus cloud of greater or less extent, or even sometimes to a large dark cloud if the extent and duration of the whirlwind are sufficiently great. As the condensed vapor is carried up in the shape of a cumulus cloud to a considerable height above its base, and the progressive velocity of the upper strata is greater than that of the lower, the tops are blown forward in the general direction of the currents, so that they often appear of an oblong shape and indicate the general direction of the currents at that altitude. Such clouds usually appear for awhile and gradually vanish by the re-evaporation of the condensed vapor forming them.

Pumping of the barometer.

262. In connection with frequent changes of the velocity, and consequently force of the wind, there is an unsteadiness of barometrical column called "pumping," where the barometer is placed on one side or the other of a barrier to the progress of the air. This effect is given in millimeters of the barometer by (15), § 226. According to this expression, where the barometer is placed against a wall or post where the wind blows normally against its surface with a velocity of 10 meters per second, the height of the barometer is increased 0.5^{mm}. If the velocity is increased to 20 meters per second it becomes 2.0^{mm}, and hence a change of 1.5^{mm} with a change of velocity from 10^{m} to 20^{m} per second.* With a change, however, from 20 to 30^{m} per second the change in barometric pressure would be 2.5^{mm}. Hence the same change of velocity where the velocity is already great gives a much greater effect upon the barometer than the same change in velocity where the velocity as yet is small. Hence in cyclones where the general velocity is great, small changes in velocity produce a considerable effect on the barometer, and it is in cyclones, therefore, where this effect is mostly observed.

Much depends upon the position of the barometer. If placed in the open air little or no effect is observed, since there is little obstruction to the wind. If placed where wind blows obliquely against the face of the barrier, it is seen from the effect of the factor $\cos^2 i$ in the formula that the effect is diminished in proportion to $\cos^2 i$ of the angle of incidence of the wind. On the lee side of a barrier there is a slight depression of the barometer with the increase of velocity in the blasts arising from the dragging away of the air on that side through friction. A barometer placed in a tight room of course cannot be much affected, and perhaps not sensibly in any room with doors and windows closed, especially when the blasts are sudden and of so short duration that there is not time for the increase of pressure to be felt inside.

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* These effects were given erroneously in Meteorological Researches, part II, § 122.

In the hurricanes of the Antilles observation shows that these small oscillations of the barometer are closely connected with, and dependent upon, the blasts of the wind and that oscillations of the vane always accompany the blasts, showing that the latter are due to small gyrations of the air. Padre Viñes says[12]:

> Under the influence of the blasts the barometric column is so agitated and so irregular that it renders the reading of it very difficult, since it is scarcely possible to take an exact medium. The amplitude of the oscillations is usually from four to eight tenths of a millimeter, and sometimes more. The agitation is fitful and violent, just as the impulses which the anemometer receives and the oscillations made by the vane.

Mackerel sky.

263. Each cloudlet in a mackerel sky arises from the condensation of vapor carried up to a higher altitude in a little whirlwind in the cloud-regions when the atmosphere is in the unstable state, and saturated, or nearly so, with aqueous vapor. At considerable altitudes above the earth's surface the unstable state for saturated air is very nearly the normal or average state, at least in the summer season. Whenever, therefore, the air at these altitudes becomes very nearly saturated, and from some slight cause becomes changed a little from its normal condition into that of the unstable state for saturated air, it then bursts up at numerous places through the strata above, with generally a slight whirl, and the vapor carried up in the central part of each one of these to a little higher altitude, where the air is colder, is condensed into a little cumulus cloud, just as in the case of large cumulus clouds nearer the earth's surface. As the mackerel sky is produced only when the atmosphere above is nearly saturated with aqueous vapor and in the unstable state, it is an indication of rainy weather.

CHAPTER VI.

METEOROLOGICAL OBSERVATIONS AND THEIR REDUCTIONS.

Harmonic analysis.

264. The harmonic analysis is applicable, not only to vibrations and wave oscillations of different periods existing together, but likewise to any meteorological quantity, as temperature, barometric pressure, &c., which is a function of the time t, and contains regular periodic inequalities, which can be expressed under the form of

$$(1) \qquad y=\Sigma A_s \cos (i_s t-\varepsilon_s)$$

in which s equal 0, 1, 2, &c., is the characteristic of the inequality, and in which $i_0=0$, and $\varepsilon_0=0$. The first term in the series, therefore, becomes simply the constant A_0. The function is expressed here in terms of the cosine, after the French method and that adopted by Thomson and Tait,[1] and generally used in tidal analyses, instead of those of the sine, as in Bessel's formula.

The notation adopted is that of Thomson and Tait, in which for any term of which s is the characteristic.

A_s=the coefficient of the inequality or component, called the amplitude.

ε_s=the time expressed in arc from the origin of t, or time $t=0$, to the time of the maximum of the inequality, and called the epoch;

i_s=the rate with which the angle changes.

In the most general expression of y the values of i for the different components may have any relations to one another whatever; but in meteorological expressions the value of i in any of the inequalities is, with few exceptions, a multiple of that in the first and principal inequality, to which is assigned the characteristic $s=1$, and, including the first and constant term, they have the relations of 0, 1, 2, 3, &c. Where the expression of (1) is confined to these simple relations, as it usually is in meteorology, it can be put into the following form:

$$(2) \qquad y=\Sigma\ A_s \cos (si_1 t-\varepsilon_s),$$

in which i_1 is the value of i belonging to the first and principal inequality. This expression in meteorology is always convergent, and generally so much so that it is necessary to use only a few terms in the series.

The object of the analysis is to determine from observations of y, or from averages of observations, for given values of the angle si_1t, generally taken at times which differ by some submultiple of the period, the most probable values of A and ε for each of the inequalities, and the times of the maxima and minima of y resulting from the combination of all the components of the series.

265. If we put

$$(3) \qquad s\varphi = s i_1 t - \varepsilon_s - 2sn\pi,$$

in which n denotes 0, or any integral number, then $s\varphi$ is called the *phase* of the component of which the characteristic is s. The relation between the period and rate of change of the phase is given by

$$(4) \qquad T = \frac{2\pi}{i}$$

in which T is the period of the inequality

If in (3) we put

$$(5) \qquad \theta = i_1 t - 2n\pi$$

we have

$$(6) \qquad s\varphi = s\theta - \varepsilon_s$$

in which the origin of 0 is that of t, and is entirely arbitrary. If we now put

$a =$ the average of all the observations of y between the limits θ' and θ'' of the angle of the principal component,

and the observations are sufficiently numerous to eliminate all sensible effects of the abnormal influences, and are equally distributed between the limits, we shall have

$$(7) \qquad a = \frac{\int y d\theta}{\theta'' - \theta'}$$

By means of (3), (4), and (5), neglecting the constant $2sn\pi$, since being a multiple of 2π it does not affect the value of the cosine, the expression of (2) becomes

$$(8) \qquad y = \Sigma A_s \cos s\varphi = \Sigma A_s \cos (s\theta - \varepsilon_s)$$

Substituting this value of y in (7) and integrating from $\theta = \theta'$ to $\theta = \theta''$, we get

$$(9) \qquad \begin{aligned} a &= \Sigma \int \frac{A_s s \cos (s\,\theta - \varepsilon_s)\, d\,(s\,\theta)}{s\,(\theta'' - \theta')} \\ &= \Sigma A_s \frac{\sin (s\,\theta'' - \varepsilon_s) - \sin (s\,\theta' - \varepsilon_s)}{s\,(\theta'' - \theta')} \end{aligned}$$

If we suppose the limits θ' and θ'' to be taken at equal intervals before and after any given value of θ, denoted by θ_s, we shall have

$$(10) \qquad \theta_s = \tfrac{1}{2}(\theta' + \theta'')$$

and denoting by a_s the value of a corresponding to θ_s, the preceding expression then becomes

$$(11) \qquad a_s = \Sigma_s k_s A_s \cos(s\theta_s - \varepsilon_s)^{*}$$

in which

$$(12) \qquad k_s = \frac{2 \sin \tfrac{1}{2}\, s\,(\theta'' - \theta')}{s\,(\theta'' - \theta')}$$

The values of k$_s$, it is seen, depend upon the range of the angle θ between the limits θ' and θ'' for which the average has been obtained.

The following table contains the values of k, and also of their reciprocals, for the given values of $s(\theta''-\theta')$:

$s(\theta''-\theta')$	15°	30°	45°	60°	90°	120°	180°
k_s	0.9972	0.9886	0.9745	0.9549	0.9003	0.8270	$\frac{2}{\pi}$=0.6366
$\frac{1}{k_s}$	1.0028	1.0115	1.0262	1.0472	1.1107	1.2092	$\frac{\pi}{2}$=1·5708

Where the range is so small that the sine may be taken for the angle itself, we have $k_s=1$, and the value of a becomes the same as the normal value of y in (2), corresponding to any given value of θ, or of t.

Where $\theta''-\theta'$ has a considerable range, as 15°, or 30°, or more, the expression of a in (11) differs sensibly from that of y in (2), and the latter is reduced to the former by multiplying each of the components by the reciprocal of k_s for that component, as given by (12). On the contrary, an expression of the form of (2) which represents y for any given value of t, does not represent the average values of y taken through a range of the time angle $\theta''-\theta'$, unless each component in the expression is multiplied into the value of k_s for that component. The reason of this is that the average of the values of y, through a considerable range of θ or of t, is less than the value of y for the middle of the range. The values of k_s and of its reciprocal above, used as a factor, correct for this difference.

In the case in which the whole period of the first and principal inequality is divided into 24 equal parts, in which case we have $\theta''-\theta'$ equal 15°, the value of k_s differs but little from unity, and the correc-

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[*] From the trigonometrical relations $\sin a - \sin b = 2 \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b)$ found in any treatise on trigonometry, putting $5\,\theta'' - \varepsilon_s$ for a and $s\,\theta' - \varepsilon_s$ for b we get $\sin(s\,\theta'' - \varepsilon_s) - \sin(s\,\theta' - \varepsilon_s) = 2 \sin \tfrac{1}{2}\, s\,(\theta'' - \theta') \cos \tfrac{1}{2}(s\,\theta' - \varepsilon_s + s\,\theta'' - \varepsilon_s)$. This by means of (10) is reduced to (11).

tion for this component is small, but for the others corresponding to $s=2$, $s=3$, &c., it is larger, for in case of the second component we should have the range $s(\theta''-\theta')$ equal 30°, and in the third equal 45°, and so on, and consequently the correcting factor, as seen from the preceding table, differs more and more from unity.

266. In order to determine the most probable values of the constants A_s and ε_s in (2) from observed values of y, affected by various abnormal disturbances and errors of observation, it is usual to take a certain number of averages, represented by a, for ranges of equal intervals comprising a whole period of the first and principal component, as a day in the case of a diurnal inequality, or a month in the case of an annual inequality, of some observed quantity as temperature or barometric pressure. In the first case we should have 24 hourly averages and the value of $\theta''-\theta'$ would be (15°), and in the latter, 12 monthly averages, in which case we should have $\theta''-\theta'$ equal 30°. All the observations of y which fall within the range of each hour for a great many days can be taken to make up the averages in the one case, and all the diurnal values of y that fall within the range of each month for a great many years can be used in the other. In fact strict accuracy would require that the observations should be so numerous that they can be regarded as being distributed somewhat evenly through the whole ranges or groups for which the averages are taken, especially when the period is divided into only a few equal parts and where, consequently, the ranges $(\theta''-\theta')$ are large.

In order to determine the constants in (2) from any set of averages or values of a in (11) it is necessary to put this expression into the following form:

$$a=\Sigma M_s \cos s\theta_s+\Sigma N_s \sin s\theta_s{}^* \tag{13}$$

in which

$$M_s = k_s A_s \cos \varepsilon_s \qquad N_s = k_s A_s \sin \varepsilon_s \tag{14}$$

For convenience, θ is used instead of $i_1 t$ in this expression, for the part $2n\pi$ in (5) does not affect the cosines and sines.

From (14) we get

$$A_s = \frac{1}{k_s}\sqrt{M^2_s+N^2_s}=\frac{M_s}{k_s \cos \varepsilon_s} \qquad \tan \varepsilon_s = \frac{N_s}{M_s} \tag{15}$$

With n values of a in (13) denoted by A_0, A_1, A_2, ... A_n, or in general by A_s obtained from the averages of observations for correspond-

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[*] Developing $\cos(s\theta-\varepsilon_s)$ into $\cos s\theta \cos \varepsilon_s+\sin s\theta \sin \varepsilon_s$, and substituting this in (11) per $\cos si_1t-\varepsilon_s=\cos(s\theta-\varepsilon_s)$ we get

$$a=\Sigma k_s A_s \cos \varepsilon_s \cos s\theta = \Sigma k_s A_s \sin \varepsilon_s \sin \theta$$
$$=\Sigma M_s \cos s\theta+\Sigma N_s \sin \theta$$

ing values of θ or $i_1 t$, denoted by θ_0, θ_1, θ_2, ... θ_n, or in general by θ_e, (13) furnishes as many equations of conditions for determining by the method of least squares the most probable values of M_s and N_s, and then with these and the values of k_s obtained from (12) or the preceding small table, we get from (15) the values of A_s and ε_s the required constants in (2). With these constants (2) then gives the value of y for any given time t.

267. The solution by the method of least squares becomes very much simplified where the averages are taken for equal intervals of the period, but this is not essential provided the averages or values of a in (13) are first corrected for each group of varying range, but then we must put $k_s=1$ in (14) and (15). Where the intervals are all equal we get from (13) n equations of the general form.

(16) $$a_e = \Sigma_s M_s \cos \theta_e + \Sigma_s N_s \cos \theta_e$$

in which e has the values 0, 1, 2, &c., up to n, the number of average values in the period, thus giving n equations, with terms in each corresponding to the values of s equal 0, 1, 2, &c., according to the number of terms it is thought proper to include. For instance, the preceding general equation written out in detail for any one of the groups of observations contained in the average, of which the characteristic is $e=3$, would be

(17) $$a_3 = M_0 + M_1 \cos \theta_3 + N_1 \sin \theta_3 + M_2 \cos 2\theta_3 + N_2 \sin 2\theta_3$$
$$+ M_3 \cos 3\theta_3 + N_3 \sin 3\theta_e \; . \; . \; . \; . \; . \; .$$

In like manner each of the n equations in the general expression of (16) can be written. If we now take the terms in order and multiply each of the terms in the n equations of (16) of the form of (17) by the cosine or sine, which is the coefficient of the unknown quantity to be determined, and add all the resulting n equations together, as is usual in the method of least squares, in order to obtain as many normal equations, these equations become very much simplified by the following well-known relations:

(18) $$\Sigma_e \cos (s\theta_e + c) = 0 \qquad \Sigma_e \cos^2 (s\theta_e + c) = \tfrac{1}{2}n$$

in which c is any arbitrary constant, and may be assumed either equal 0, or $\frac{1}{2}\pi$; in the former case these general expressions give

(19) $$\Sigma_e \cos s\theta_e = 0 \qquad \Sigma_e \cos^2 s\theta_e = \tfrac{1}{2}n$$

and in the latter

(20) $$\Sigma_e \sin s\theta_e = 0 \qquad \Sigma_e \sin^2 s\theta_e = \tfrac{1}{2}n$$

The products of sines and cosines into one another in the multiplications are readily reduced by means of well-known trigonometrical relations to two others of the forms of the first of (19) or (20), and hence the sums of all such terms arising from adding together the n equations vanish, and there is left only the sum of the one column containing $\cos^2 \theta_s$ or $\sin^2 \theta_s$, as the case may be. Hence, in carrying the process through in the usual way, we get as many normal equations as there are unknown constants M_s and N_s, which by the relations of (19) and (20) are reduced to the following general form of expression:

$$(21) \qquad M_s=\frac{2}{n}\Sigma_s a_s \cos s\theta_s \qquad\qquad N_s=\frac{2}{n}\Sigma_s a_s \sin s\theta_s$$

In the case of the constant M_0, however, we get by adding together all the n equations and reducing by the first of (19) or (20)

$$(22) \qquad M_0=\frac{\Sigma a_s}{n}$$

268. In meteorology, for the most part, only three cases of these general expressions are used: That in which $n=8$, as in tri-hourly observations; that in which $n=12$, as in the case of monthly averages; and that in which $n=24$, as in the case of hourly averages. We shall, therefore, write out these general expressions in detail for these three cases, for convenience in the practical applications of them.

In the case in which $n=8$, we get from (21) and (22)

$$8M_0=a_0+a_1+a_2 \quad . \quad . \quad . \quad a_7$$
$$4M_1=a_0 \cos 0^\circ+a_1 \cos 45^\circ+a_2 \cos 90^\circ \quad . \quad . \quad . \quad . +a_7 \cos 315^\circ$$
$$4M_2=a_0 \cos 0^\circ+a_1 \cos 90^\circ+a_2 \cos 180^\circ \quad . \quad . \quad . \quad . +a_7 \cos 270^\circ$$
$$4M_3=a_0 \cos 0^\circ+a_1 \cos 135^\circ+a_2 \cos 270^\circ \quad . \quad . \quad . \quad . +a_7 \cos 225^\circ$$

. . . .

$$4N_1=a_0 \sin 0^\circ+a_1 \sin 45^\circ+a_2 \sin 90^\circ \quad . \quad . \quad . \quad . +a_7 \sin 315^\circ$$
$$4N_2=a_0 \sin 0^\circ+a_1 \sin 90^\circ+a_2 \sin 180^\circ \quad . \quad . \quad . \quad . +a_7 \sin 270^\circ$$
$$4N_3=a_0 \sin 0^\circ+a_1 \sin 135^\circ+a_2 \sin 270^\circ \quad . \quad . \quad . \quad . +a_7 \sin 225^\circ$$

In these expressions some of the cosines and sines become equal to unity and others vanish. They may therefore be put into the following more convenient form for practical application:

$$8M_0=\Sigma a_e$$

$$4M_1=(a_0-a_4)+\begin{pmatrix} a_1-a_5 \\ a_7-a_3 \end{pmatrix} \cos 45^\circ$$

$$4N_1=(a_2-a_6)+\begin{pmatrix} a_1-a_5 \\ a_3-a_7 \end{pmatrix} \sin 45^\circ$$

$$4M_2=\begin{pmatrix} a_0-a_2 \\ a_4-a_6 \end{pmatrix}$$

$$4N_2=\begin{pmatrix} a_1-a_3 \\ a_5-a_7 \end{pmatrix}$$

$$4M_3=(a_0-a_4)+\begin{pmatrix} a_3-a_7 \\ a_5-a_1 \end{pmatrix} \cos 45^\circ$$

$$4N_3=(a_6-a_2)+\begin{pmatrix} a_3-a_7 \\ a_1-a_5 \end{pmatrix} \sin 45^\circ$$

269. In the case in which $n=12$, we get from (21) and (22)

$$12M_0=a_0+a_1+a_2 \quad . \quad . \quad . \quad . \quad . \quad +a_{11}$$

$$6M_1=a_0+a_1 \cos 30^\circ+a_2 \cos 60^\circ \quad . \quad . \quad . \quad . \quad +a_{11} \cos 330^\circ$$

$$6M_2=a_0+a_1 \cos 60^\circ+a_3 \cos 120^\circ \quad . \quad . \quad . \quad . \quad +a_{11} \cos 300^\circ$$

$$6M_3=a_0+a_1 \cos 90^\circ+a_3 \cos 180^\circ \quad . \quad . \quad . \quad . \quad +a_{11} \cos 270^\circ$$

. . . .

$$6N_1=a_1 \sin 30^\circ+a_2 \sin 60^\circ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .+a_{11} \sin 330^\circ$$

$$6N_2=a_1 \sin 60^\circ+a_2 \sin 120^\circ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .+a_{11} \sin 300^\circ$$

$$6N_3=a_1 \sin 90^\circ+a_2 \sin 180^\circ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .+a_{11} \sin 270^\circ$$

In these expressions, as in the preceding, some of the cosines and sines vanish and others become equal to unity. They may therefore be put into the following more convenient form:

$$12\,M_0=\Sigma a_e$$

$$6M_1=(a_0-a_6)+\begin{pmatrix} a_1-a_7 \\ a_{11}-a_5 \end{pmatrix}\cos 30^\circ+\begin{pmatrix} a_2-a_8 \\ a_{10}-a_4 \end{pmatrix}\cos 60^\circ$$

$$6N_1=(a_3-a_9)+\begin{pmatrix} a_1-a_7 \\ a_5-a_{11} \end{pmatrix}\sin 30^\circ+\begin{pmatrix} a_2-a_8 \\ a_4-a_{10} \end{pmatrix}\sin 60^\circ$$

$$6M_2=\begin{pmatrix} a_0-a_3 \\ a_6-a_9 \end{pmatrix}+\begin{pmatrix} a_1-a_4 \\ a_7-a_{10} \\ a_5-a_8 \\ a_{11}-a_2 \end{pmatrix}\cos 60^\circ$$

$$6N_2=\begin{pmatrix} a_1-a_4 \\ a_7-a_{10} \\ a_8-a_5 \\ a_2-a_{11} \end{pmatrix}\sin 60^\circ$$

$$6M_3=\begin{pmatrix} a_0-a_{10} \\ a_4-a_2 \\ a_8-a_6 \end{pmatrix}$$

$$6N_3=\begin{pmatrix} a_1-a_3 \\ a_5-a_7 \\ a_9-a_{11} \end{pmatrix}$$

$$6M_4=\begin{pmatrix} a_0+a_3 \\ a_6+a_9 \end{pmatrix}-\begin{pmatrix} a_1+a_2 \\ a_4+a_5 \\ a_7+a_8 \\ a_{10}+a_{11} \end{pmatrix}\cos 60^\circ$$

$$6N_4=\begin{pmatrix} a_1-a_2 \\ a_4-a_5 \\ a_7-a_8 \\ a_{10}-a_{11} \end{pmatrix}\sin 60^\circ$$

270. In the case in which $n=24$ we have $\theta_0=0$, $\theta_1=15^\circ$, $\theta_2=30^\circ$, &c., and hence the expressions of $12M_s$ and $12N_s$ can be written out as above

from the general equations of (21), but they will contain twice as many terms. These, then, can be arranged in the following form for the convenience of practical application:

$$24M_0=\Sigma_e\, a_e$$

$$12M_1=(a_0-a_{12})+\begin{pmatrix}a_1-a_{13}\\a_{23}-a_{11}\end{pmatrix}\cos 15^\circ+\begin{pmatrix}a_2-a_{14}\\a_{22}-a_{10}\end{pmatrix}\cos 30^\circ$$
$$+\begin{pmatrix}a_3-a_{15}\\a_{21}-a_9\end{pmatrix}\cos 45^\circ+\begin{pmatrix}a_4-a_{16}\\a_{20}-a_8\end{pmatrix}\cos 60^\circ+\begin{pmatrix}a_5-a_{17}\\a_{19}-a_7\end{pmatrix}\cos 75^\circ$$

$$12N_1=a_6-a_{18}+\begin{pmatrix}a_1-a_{13}\\a_{11}-a_{23}\end{pmatrix}\sin 15^\circ+\begin{pmatrix}a_2-a_{14}\\a_{10}-a_{22}\end{pmatrix}\sin 30^\circ$$
$$+\begin{pmatrix}a_3-a_{15}\\a_9-a_{21}\end{pmatrix}\sin 45^\circ+\begin{pmatrix}a_4-a_{16}\\a_8-a_{20}\end{pmatrix}\sin 60^\circ+\begin{pmatrix}a_5-a_{17}\\a_7-a_{19}\end{pmatrix}\sin 75^\circ$$

$$12M_2=\begin{pmatrix}a_0-a_6\\a_{12}-a_{18}\end{pmatrix}+\begin{pmatrix}a_1-a_7\\a_{13}-a_{19}\\a_{11}-a_{17}\\a_{23}-a_5\end{pmatrix}\cos 30^\circ+\begin{pmatrix}a_2-a_8\\a_{14}-a_{20}\\a_{10}-a_{16}\\a_{22}-a_4\end{pmatrix}\cos 60^\circ$$

$$12N_2=\begin{pmatrix}a_3-a_9\\a_{15}-a_{21}\end{pmatrix}+\begin{pmatrix}a_1-a_7\\a_{13}-a_{19}\\a_{17}-a_{11}\\a_5-a_{23}\end{pmatrix}\sin 30^\circ+\begin{pmatrix}a_2-a_8\\a_{14}-a_{20}\\a_{16}-a_{10}\\a_4-a_{22}\end{pmatrix}\sin 60^\circ$$

$$12M_3=\begin{pmatrix}a_0-a_4\\a_8-a_{12}\\a_{16}-a_{20}\end{pmatrix}+\begin{pmatrix}a_1-a_5\\a_9-a_{13}\\a_{17}-a_{21}\\a_7-a_{11}\\a_{15}-a_{19}\\a_{23}-a_3\end{pmatrix}\cos 45^\circ$$

$$12N_3=\begin{pmatrix}a_2-a_6\\a_{10}-a_{14}\\a_{18}-a_{22}\end{pmatrix}+\begin{pmatrix}a_1-a^5\\a_9-a_{13}\\a_{17}-a_{21}\\a_{11}-a_7\\a_{19}-a_{15}\\a_3-a_{23}\end{pmatrix}\sin 45^\circ$$

$$12M_4=\begin{pmatrix}a_0-a_3\\a_6-a_9\\a_{12}-a_{15}\\a_{18}-a_{21}\end{pmatrix}+\begin{pmatrix}a_1-a_4\\a_7-a_{10}\\a_{13}-a_{16}\\a_{19}-a_{22}\\a_5-a_8\\a_{11}-a_{14}\\a_{17}-a_{20}\\a_{23}-a_2\end{pmatrix}\cos 60^\circ$$

$$12N_4=\begin{pmatrix}a_1-a_4\\a_7-a_{10}\\a_{13}-a_{16}\\a_{19}-a_{22}\\a_8-a_5\\a_{14}-a_{11}\\a_{20}-a_{17}\\a_2-a_{23}\end{pmatrix}\sin 60^\circ$$

Of the many forms into which the preceding formulæ can be put for convenience of practical application those given here seem the best. A great advantage is that the different sets of differences are between values of a_e with a constant difference between the suffixed characteristic numbers e, so that all the values of a_e being written in order in a column on two slips, these can be readily so adjusted with a little sliding up or down as to obtain conveniently at the start all the differences required, and these being generally small in comparison with the original values of a_e, the constants and numerical coefficients of cos. or sin. can be readily obtained. The same differences also occur in the coefficients of cos. and sin., only that the signs of one-half are reversed in the latter, so that if each half is computed separately the sum of them is to be used for the coefficient of cos. and the difference for that of sin. The great amount of time and labor saved by convenience of arrangement makes it a matter of much importance.

271. The first of the relations of (18) which have been used in the reductions of the normal equations in the solution by the method of least squares, can be deduced from (9) and (11). By (11) we have, considering only one term under the sign Σ_s,

$$a_e = k_s A_s \cos(s\theta_e - \varepsilon_s)$$

in which k_s has its value corresponding to any given limits $\theta'' - \theta' = \frac{2\pi}{n}$. We shall therefore have for the sum of all the n values of A_e, the integral of (9) taken from $\theta = 0$ to $\theta = 2\pi$, and hence for any one term under the sign Σ_s, by putting for k_s its value in (12) corresponding to the new limits, and also for θ_s its value π as given by (10) with these limits, we get

$$\Sigma a_e = k_s A_s \Sigma_e \cos(s\theta_e - \varepsilon_s) = \frac{\sin s\pi}{s\pi} \cos(s\pi - \varepsilon_s) = 0$$

Hence $\Sigma_e \cos(s\theta_e - \varepsilon_s) = 0$, which is the same as the first of (18) since the epoch ε_s is entirely arbitrary and may put equal $-c$. We also have

$$\cos^2(s\theta_e - \varepsilon_s) = \tfrac{1}{2} + \tfrac{1}{2}\cos 2(s\theta_e - \varepsilon_s)$$

and hence

$$\Sigma_e \cos^2(s\theta_e - \varepsilon_s) = \Sigma_e \tfrac{1}{2} + \Sigma_e \tfrac{1}{2}\cos 2(s\theta_e - \varepsilon_s)$$

The integration of the second member through n equal intervals gives $\frac{1}{2}n$, since the integration of the last term by the relation just obtained gives 0. For $\cos 2(s\theta_e - \varepsilon_s)$ can be put under the form $\cos(2s\theta_e - 2\varepsilon_s)$, which is of the same general form of 18, since s can have any integral value. We therefore get $\Sigma_e \cos^2(s\theta - \varepsilon_s) = 0$, which, from what has been stated, is virtually the same as (18_2).

There is one exceptional case in which the relations of (19_2) and (20_2) do not hold, and that is, where there are only two intervals in a period. In this case we have only the two values of $s\theta_e$, namely, $s\theta_0 = 0$ and $s\theta_1 = 180^\circ$, and hence the sum of the cosines is $n = 2$ instead of $\frac{1}{2}n$ or unity. In this case it is also evident that the sum of the sines is 0, and not $\frac{1}{2}n=1$.

272. Where there are only four averages or values of a_s to the period of the first and principal component, as where the averages of winter, spring, summer, and autumn are taken, we get from (21), n in this case being equal 4,

$$(23) \qquad M_1=\tfrac{1}{2}(a_0-a_2) \qquad\qquad N_1=\tfrac{1}{2}(a_1-a_3)$$

But for the second or semi-annual component, since in this case there are only two intervals to the period, we have in (21_1) $\Sigma_s \cos s\theta_s=n$ instead of $\frac{1}{2}n$, as shown above, and hence

$$(24) \qquad M_2=\begin{pmatrix} a_0-a_1 \\ a_2-a_3 \end{pmatrix} \qquad\qquad N_2=0$$

We have, therefore, the four observed values of a_s to determine the four unknown constants $M_0=\Sigma a_s$, M_1, N_1, and M_2. From (15) we therefore get in this case

$$(25) \qquad A_1=\frac{1}{k_1}\sqrt{M^2{}_1+N_1}=\frac{M_1}{k_1\cos\varepsilon_1} \qquad\qquad \tan\varepsilon=\frac{N_1}{M_1}$$

The value of $1:k_1$ in this case is given in the table of § 265 in the column under 90°, this being the value of $(\theta''-\theta')$ the range of the groups of observations. By the preceding expressions (23) and (25) the annual inequality, or the first and principal of any series of inequalities of (2), can be obtained from the four groups of observations, but for the determination of any of the subordinate inequalities more groups and more equations of condition are needed. There being four groups and four equations based upon them, values of the four unknown quantities M_0, M_1, N_1, and M_2 can be determined which will make the expression represent accurately the observations.

273. With the values of M_s and N_s given by the preceding formulæ, and the values of k_s given by (12) or found in the preceding table, we get from (15) the values of A_s and ε_s, in (2), with which we can then compute the value of y for any given time t. From (11) and (12) we can likewise compute with these constants the n values of a_s used in the determination of these constants as a test, in some measure, of the accuracy of the work, though it is not in general a complete one, unless as many constants have been determined as there are observed values of a_s. Where this is not the case there will be of course small residuals not represented by the expression of (11). By a judicious inspection of these, however, we can generally detect even very small errors in the work, since where the residuals are merely accidental, depending upon uneliminated abnormal irregularities in the averages, there cannot be any regular alternating of signs in the residuals, such as is usually observed where there is an error in the constants or in the computations.

The probable errors of the constants are determined by the usual expressions deduced from the theory of least squares, which are given here for convenience of reference.

If we put

v_s=residual corresponding to any average value a_s;
ε=the probable error of the average, regarded as a single observation;
r=the probable error of the constants A_0, M_s, and N_s;
m=the number of values of a_s;
n=the number of constants determined,

we have

$$\varepsilon=0.6745\sqrt{\frac{\Sigma v_s}{m-n}} \tag{26}$$

The weight of each of the constants, as deduced from the normal equations for determining the constants, is n for A_0 and $\frac{1}{2}n$ for each of the others. Hence we have for the probable error of A_0

$$r=\frac{\varepsilon}{\sqrt{n}} \tag{27}$$

and for M_s and N_s, and consequently for A_s,

$$r=\frac{\varepsilon}{\sqrt{\frac{1}{2}n}} \tag{28}$$

The probable error of the epochs ε_s, expressed in degrees, is

$$r'=\frac{\varepsilon}{\sqrt{\frac{1}{2}n}}\times\frac{57.3^\circ}{A_s} \tag{29}$$

274. In determining by the preceding method of least squares the constants in (11) by means of a certain number of observed values of a_s at regular and equal intervals of the period of the principal inequality, the question of how many components should be taken into account becomes an important one. The values of a_s, however many observations may be included in each average, must still be regarded as being affected in some measure by the uneliminated effects of abnormal disturbance and small errors of observation, which should be excluded from the expression as much as possible and thrown into the residuals. If we take in so many components that we have as many unknown constants to determine as there are observed values of a_s and equations of condition, of course with these constants (11) becomes an expression which takes in all these uneliminated abnormal effects and all the errors of observations, and we get often an expression which is more erroneous than if we had taken into account only a very few components. And this is especially the case where the values of a_s depend upon few observations, or where there are very large abnormal effects to be eliminated by the number of observations used. We have in this

case also no means of determining probable errors, since we have in (26) $\Sigma v_s = 0$ and $m - n = 0$, and consequently the value of ε indeterminate.

Where theory determines the number of sensible components in the expression, of course these only are to be considered and the most probable constants determined for these alone. The residuals then, however large, necessarily belong to the errors of observation and the uneliminated abnormal disturbances. In meteorology, however, theory does not determine precisely the number of sensible terms, though it may aid us in judging of the convergency of the expression, and the number of components which should be taken into account must in all cases depend in some measure upon the exercise of good judgment.

Where the constants determined for a considerable number of components make the expression rapidly convergent for a few of the first components, and after that the amplitudes obtained are small and of about the same order, we are pretty safe in assuming that these small components represent merely the abnormal uneliminated effects, and should therefore be excluded from the expression as being not only useless, but even injurious. Where, also, the constants in (11) are obtained in numerous cases under circumstances which are so nearly the same that we have good reason to think that the relations between these constants should not differ very much, and they are found not only to differ very much in the case of the very small components, where a number of them have been taken into account, and without any apparent law, we may conclude that they belong to the abnormal uneliminated disturbances, and should be excluded. For instance, if the constants are determined for the temperature or barometric observations of a considerable number of stations in the same vicinity, or over a region of no very great extent, or for differen[illegible] series of years at the same station, and the epochs ε_s for the smaller terms seem to be entirely accidental and fall at all places within the period, or circumference of a circle when the epoch is expressed in degrees, we may feel assured that the components represent merely the uneliminated irregularities which should be excluded. It frequently happens that the values of ε_s for a few of the principal components are very nearly the same at all such stations and for different series of observations at the same station, but for the other small terms it may have any value within the whole circumference of a circle. If the values seem to fall a little more in one part of the circle than the opposite, it indicates that there is a very small component of the period in question, but too small to be brought out clearly by the observations. A change of 180° in the value of the epoch is equivalent to a reversal of the sign of the coefficient of the term, and hence where the epoch is as liable to fall in one part of the circumference as another, it is the same as where a term is as liable to have a minus as a plus sign, and it indicates that there is no real term of the period in question.

Maxima and minima.

275. If we put

τ=the time of the maximum of any component of (2), of which the characteristic is s,

we then have, where ε_s and i are expressed in arc in terms of the radius,

$$\tau=\frac{\varepsilon_s\pm 2sn\pi}{si} \tag{30}$$

in which $n=0$, or some even integral number for the maxima and some odd number for the minima. Where only one period is considered it is either 0 or unity. Where ε_s and i are expressed in degrees, we must put $\pi=180^\circ$. These expressions give the time from some fixed origin of τ, as some given year, or as the beginning of the year, in the case of diurnal inequalities. It is, however, usual to assume the era or origin of t to be the beginning of each year, or of each day in the case of diurnal inequalities, and we have then in (30) either $n=0$ or $n=1$.

If we put

T=the time of maximum or minimum of y, that is, of the resultant of all the components in (2):

then the condition for determining their maxima and minima is that we shall have at the time $\mathrm{D}y=0$ in (2).

Taking the differential, and putting by way of abbreviation φ_s for $si_1T-\varepsilon_s$ after differentiation, we get

$$0=\Sigma sA_s\sin s\varphi=\Sigma sA_s\sin\left[\varphi_1+(\varphi_s-\varphi_1)\right]$$

$$=\Sigma sA_s\cos(\varphi_s-\varphi_1)\sin\varphi_1+\Sigma sA_s\sin(\varphi_s-\varphi_1)\cos\varphi_1$$

$$=M\sin\varphi_1+N\cos\varphi_1$$

in which

$$M=\Sigma sA_s\cos(\varphi_s-\varphi_1) \qquad N=\Sigma sA_s\sin(\varphi_s-\varphi_1)$$

The preceding expression can be put into the following form:*

$$0=R\sin(\varphi_1+\beta)=R\sin(i_1T-\varepsilon_1+\beta) \tag{31}$$

in which

$$R=\sqrt{M^2+N^2}=\frac{M}{\cos\beta}=\frac{N}{\sin\beta}$$

$$\tan\beta=\frac{N}{M} \tag{32}$$

$$M=\Sigma sA_s\cos\left[(s-1)\,i_1T-\varepsilon_s+\varepsilon_1\right]$$

$$N=\Sigma sA_s\sin\left[(s-1)i_1T-\varepsilon_s+\varepsilon_1\right]$$

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[*] We can put $M\sin\varphi_1+N\cos\varphi_1=R\cos\beta\sin\varphi_1+R\sin\beta\cos\varphi_1$ in which $M=R\cos\beta$, $N=R\sin\beta$, and hence R and $\tan\beta$ as in (32). But the last member of the preceding equation, by a well-known trigonometrical relation, is expressed by $R\sin(\varphi_1+\beta)$.

For the first terms in the expressions of M and N, since for these terms $s=1$, we have the angle equal 0, and hence the first term of M is A, and the first of N is equal 0. As the first inequality in the expression of (2) is usually large in comparison with the others, the value of M is much larger than N, and hence the value of the angle β in (32_2) is usually small. At the times of maxima and minima we must have by (31)

$$i_1 T - \varepsilon_1 + \beta = n\pi$$

in which n equals 0 or some even integral number for the maxima, and some odd one for the minima. But usually T is reckoned from the beginning of the year or day, and then we have $n=0$ for the maximum and $n=1$ for the minimum next following.

This gives

$$(33) \qquad T = \frac{\varepsilon_1 - \beta \pm n\pi}{i_1}$$

In the computations of β from (32) with which to obtain T from (33), the value of T is required in the formula, and hence it is necessary to know at the start the thing sought. The value of T, therefore, in (33) can only be obtained by approximations, but this is readily done, since β is usually so small that the uncertainty in the time to be assumed in the first approximation is so little that even a second approximation is often not necessary, and rarely a third one, if the values be judiciously assumed at the start.

If we suppose β is so small that we can put, without sensible error, $\tan \beta = \beta$, and neglect all the small terms in the expression of M (32_3), and retain only the first, which we have seen becomes A_1, we can put for the time of maximum or minimum,

$$(34) \ \beta = \frac{N}{M} = \frac{2A_2 \sin(i_1 T - \varepsilon_2 + \varepsilon_1) + 3A_3 \sin(2i_1 T - \varepsilon_3 + \varepsilon_1) +, \&c.}{A_1}$$

in which T is used for t.

As the values of A_2, A_3, &c., are generally small in comparison with A_1, the value of β is so small that with the known value of ε_1 in (33), and with an approximate value of β, obtained by mere inspection of (34), or from a rough preliminary computation with some assumed approximate value of T, a value of T to be used in (32) in computing M, and N, can usually be obtained which needs no further correction. If, however, the value of β given by (32_2) differs much from the value obtained from (34) with the assumed value, another approximation is necessary.

276. In order to determine at what time the value of y in (2) is equal to the mean, putting, by way of abbreviation, $\varphi_s = s i_1 t - \varepsilon_s$, we can put (2) into the following form:

$$(35) \quad \begin{aligned} y - A_0 &= \Sigma A_s \cos \varphi_s = \Sigma A_s \cos\left[\varphi_1 + (\varphi_s - \varphi_1)\right] \\ &= \Sigma A_s \cos \varphi_1 \cos(\varphi_s - \varphi_1) - \Sigma A_s \sin \varphi_1 \sin(\varphi_s - \varphi_1) \\ &= M \cos \varphi_1 - N \sin \varphi_1 = R \cos(\varphi_1 + \beta) \end{aligned}$$

in which

$$R=\sqrt{M^2+N^2}=\frac{M}{\cos\beta}=\frac{N}{\sin\beta}$$

$$(36)\qquad \tan\beta=\frac{N}{M}$$

$$M=\Sigma A_s\cos(\varphi_s-\varphi_1)=\Sigma A_s\cos\left[(s-1)\,i_1t-\varepsilon_s+\varepsilon_1\right]$$

$$N=\Sigma A_s\sin(\varphi_s-\varphi_1)=\Sigma A_s\sin\left[(s-1)\,i_1t-\varepsilon_s+\varepsilon_1\right]$$

In these expressions of M and N the constant A_0 does not enter, since it is transferred to the first member above.

When $y=A_0$, which is its mean value, we have, putting R' and β' for values of R and β in this case, respectively,

$$(35')\qquad 0=R'\cos(\varphi_1+\beta')$$

in which

$$R=\sqrt{M^2+N^2}$$

$$(36')\qquad \tan\beta'=\frac{N}{M}$$

$$M=\Sigma A_s\cos\left[(s-i)\,i_1\,T'-\varepsilon_s+\varepsilon_1\right]$$

$$N=\Sigma A_s\sin\left[(s-1)\,i_1\,T'-\varepsilon_s+\varepsilon_1\right]$$

in which T' is put for the value of t when $y=A_0$.

But (35′) can only be satisfied with

$$i_1\,T'-\varepsilon_1+\beta'=\pm\tfrac{1}{2}n\pi$$

in which n equals some odd integral number. We therefore get

$$(37)\qquad T'=\frac{\varepsilon_1-\beta'\pm\frac{1}{2}n\pi}{i_1}$$

For the first and principal term in the expressions of M and N in (36) we have $s=1$, and hence the angle equal 0. The first term in the expression of M, therefore, is A_1, and the first one in that of N vanishes, and as all the other terms are comparatively small generally, the value of β_1 is small. Where this is so small that we can put $\tan\beta=\beta$, we have

$$(38)\qquad \beta'=\frac{N}{M}=\frac{A_2\sin(i_1T'-\varepsilon_2+\varepsilon_1)+A_3\sin(2i_1T'-\varepsilon_3+\varepsilon_1)+,\ \&c.}{A_1}$$

This, as (34) in the preceding case, serves to get the first approximate value of B to be used in (37) in computing T', and with this (36′) then gives M and N and $\tan\beta$. If this gives a value of β differing much from that obtained from (38) with an assumed value of T', another approximation is necessary.

277. If the values of y in (2), after the constants have been obtained, are computed for certain equal and small intervals of the period of the principal component, and the differences, $\varDelta_1, \varDelta_2, \varDelta_3$, &c., are arranged as in the examples which follow, the values of T or of T' may be obtained from the usual interpolation formula.

$$(39) \qquad y = y_0 + T\varDelta_1 + \frac{T(T-1)}{2}\varDelta_2 + \frac{(T+1)T(T-1)}{2.3.}\varDelta_3 +, \&c.$$
$$= y_0 + B_1T + \tfrac{1}{2}B_2T^2 + \tfrac{1}{3}B_3T^3$$

in which y_0 is the value of y where τ is assumed equal 0, and in which

$$(40) \qquad \begin{aligned} B_1 &= (\varDelta_1 - \tfrac{1}{2}\varDelta_2) - \tfrac{1}{6}(\varDelta_3 - \tfrac{1}{2}\varDelta_4) +, \&c. \\ B_2 &= \varDelta_2 - \tfrac{1}{6}\varDelta_4 +, \&c. \\ B_3 &= \tfrac{1}{2}(\varDelta_3 - \tfrac{1}{2}\varDelta_4) -, \&c. \end{aligned}$$

The value of T is expressed in units of the interval between the computed values of y, and hence hours where the interval is an hour.

Taking the differential and putting $dy=0$, (39) gives at the time of maximum or minimum,

$$(41) \qquad 0 = B_1 + B_2T + B_3T^2 +, \&c.$$

Where it is necessary to use only three terms this equation gives, by the usual formula for solving quadratic equations,

$$(42) \qquad T = \tfrac{1}{2}\frac{B_2}{B_3} \pm \sqrt{\tfrac{1}{4}\left(\frac{B_2}{B_3}\right)^2 - \frac{B_1}{B_3}}$$

The value of y_0 should be taken as near as possible to the maximum or minimum, so as to make the values of T small. The value of T is to be counted from the time when $y=y_0$.

If we put T' for the time when the value of y in (2) is equal to A_0, then we must put in (39) $y=A_0$, and we thus get

$$(43) \qquad A_0 - y_0 = B_1T' + \tfrac{1}{2}B_2T'^2 + \tfrac{1}{3}B_3T'^3 +, \&c.$$

By taking y_0 as near as possible to A_0, so as to make T' small, this is obtained where it is necessary to use only two terms, by the usual formula of quadratic equations.

In all cases the equations (41) and (43) can soon be accurately solved by a few approximations.

Applications.

278. The following table contains the tri-hourly average temperatures in degrees Fahrenheit, observed at the Naval Observatory during the

years 1862–'78, inclusive, 17 years, for the three winter months, the three summer months, and the year:

Hour.	Winter months.		Summer months.		Year.				
	Observed a_e.	Computed a_e.	Observed a_e.	Computed a_e.	Observed a_e.	Computed a_e.	Differences.		
	°	°	°	°	°	°	Δ_1.	Δ_2.	Δ_3.
0	31.00	31.00	69.14	69.14	49.64	49.64	−0.49	+.10	−.18
1	....	30.51	....	68.70	....	49.15	0.57	−.08	−.07
2	....	30.08	....	68.08	....	48.58	0.72	−.15	+.15
3	29.72	29.72	67.20	67.20	47.86	47.86	0.72	.00	.34
4	....	29.87	....	66.35	....	47.14	−0.36	+.34	.40
5	....	29.04	....	66.09	....	46.76	+0.36	.74	+.20
6	28.80	28.80	66.95	66.95	47.12	47.12	1.30	.94	−.13
7	....	28.87	....	69.08	....	48.42	2.11	.81	.37
8	....	29.51	....	72.09	....	50.53	2.55	+.44	.50
9	30.97	30.97	75.25	75.25	53.08	53.08	2.49	−.06	.09
10	....	33.09	....	78.01	....	55.57	2.34	.15	−.19
11	....	35.56	....	80.12	....	57.91	−2.00	.34	.30
12	37.84	37.84	81.65	81.65	59.91	59.91	1.36	.64	−.16
13	....	39.41	....	82.75	....	61.27	+0.56	.80	−.08
14	....	39.97	....	83.38	....	61.83	−0.27	.83	+.01
15	39.53	39.53	83.35	83.35	61.56	61.56	1.11	.84	+.24
16	....	38.36	....	82.47	....	60.45	1.71	.60	.36
17	....	36.85	....	80.78	....	58.74	1.95	−.24	.33
18	35.59	35.39	78.55	78.55	56.79	56.79	1.86	+.09	.18
19	....	34.18	....	76.16	....	54.93	1.59	.27	+.02
20	....	33.28	....	73.92	....	53.34	1.30	.29	−.02
21	32.61	32.61	72.04	72.04	52.04	52.04	1.03	.27	.02
22	....	32.05	....	70.63	....	51.01	0.78	.25	.06
23	....	31.51	....	69.70	....	50.23	−0.59	+.19	−.09

With the eight observed values of a_0, a_1, a_2, &c. in this table, the formulæ of § 268 give the values of M_s and N_s, and then with these (15) gives the following values of A_s and ε_s:

Constants.	A_0.	A_1.	ε.	A_2.	ε_2.	A_3.	ε_3.	A_4,	ε_4.
	°	°	° ′	°	° ′	°	° ′	°	
Winter months............	33.23	4.841	229 20	1.834	50 40	0.460	234 44	0.025	0
Summer months...........	74.27	8.354	218 23	1.553	31 40	0.679	64 23	0.195	180
Year	58.50	6.959	221 59	1.773	37 20	0.181	78 05	0.130	180

The values of a_e being all observed at the beginning of each third hour, and not being the average of observations taken within a range of three hours, one and a half hours preceding and following, the value of k_s in this case is unity, since the range of observations is 0, and is not that given by (12) or the table following, for a range of three hours or 45°, for these are applicable in the case in which the observations are evenly distributed throughout the whole range.

There being only 8 observed values of a_s as many constants have been determined, and consequently there are no residuals in this case from which to determine probable errors, and by (26) these become indeterminate. It cannot be supposed in this case that any of the several components taken into account are insensible terms, and hence with only 8 observed values of a_s there is no basis for determining probable errors.

Since in the last component of which the characteristic is s, there are only two intervals or values of a_s to the period, we have seen, § 271, that $N_0=0$, and hence by (15) the value of ε_s is necessarily either 0 or 180°. It is not to be supposed, however, that this is really the case, but merely that with only 8 observed values of a_s we cannot determine $M_0=A_0$, and all the amplitudes and epochs of four components. With more observed values of a_s, both the amplitude and epoch of the fourth component could be determined, and also those of others, if there were other sensible components; and with 16 or 24 observed values, we would have data for determining also the probable errors of the constants.

It is seen from the preceding results that the amplitudes of the several components are much smaller and the whole expression of the diurnal inequality of the form of (2) much more convergent in winter than in summer. The indications, however, are that even in the latter season the components beyond the fourth component are extremely small, if at all sensible.

279. By (30) we get for the time of maximum after 0^h of the first and principal component, in the average of the year,

$$\tau=\frac{221^\circ.98}{15^\circ}=14^h\ 48^m$$

and the minimum is of course 12 hours preceding or following. With the values of ε_1 for winter and summer, we get τ a little greater in the former and a little less in the latter season. The time of maximum of the second component is

$$\tau=\frac{37^\circ.33}{2\times15^\circ}=1^h\ 47^m$$

And consequently at equal intervals of 12 hours before and after.

From (32) we obtain with the values of A_s and ε_s above for the average of the year, and a first assumed value of T, which is very nearly that of τ just determined, since β in (34) is generally small on account of the smallness of the other components, first approximations of M and N, and of β, which are afterwards corrected by a second approximation or more, so as to make the final assumed value of T coincide sensibly with the value given by (33). We thus get $\beta=9^\circ\ 10^\circ$. With this value (33) gives, putting $n=0$,

$$T=\frac{221^\circ.98-9^\circ.17}{15^\circ}=14^h\ 11^m$$

for the time of maximum of y in (2), or maximum of the resultant of all components. This, it is seen, from the effect of the smaller component, makes the maximum of the resultant of all the components fall a little earlier than that of the maximum of the first and principal component.

In the same manner the value of β by (32) at the time of minimum is found to be—$34^\circ\ 11'$. With this value and the value of $n=1$, we get from (33)

$$T=\frac{221^\circ.98+34^\circ.19-180^\circ}{15^\circ}=5^h\ 4^m$$

In the same manner with the values of $A_{,}$ and $\varepsilon_{,}$ for winter and summer, we find for the maximum of the winter $T=14^h\ 2^m$ and for the minimum $T=6^h\ 35^m$, and for the maximum of summer $T=14^h\ 28^m$ and for the minimum $T=4^h\ 48^m$. These being for the averages of the three months, of course the differences between the extreme values of winter and summer are somewhat greater.

With the times of maxima or minima, used for t in (2), this gives the extreme values of the temperature for the day.

In the same manner the values of β' may be obtained from (36), using (38) to get the first approximate value, and then with this value (37) gives the value of T, by putting $n=-1$, where the mean value of y in (2) is the one which precedes, and $n=1$ where it is the one which follows, the maximum.

By the method of § 277, using the differences of the computed hourly values of a, in the preceding table, for the average of the year, we get from (40) for the minimum, assuming 5^h as the time from which T is reckoned, and using the unwritten fourth order of differences, where sensible in their effect,

$$B_1=.36-.37-.03-.02=-.06,\ B_2=.74+.03=.77,\ B_3=.10+.05=.15$$

With these values of the constants (42) gives $T=.08^h=4.8^m$, and consequently the time of minimum is $5^h\ 4.8^m$, which is very nearly the same as $5^h\ 4^m$ obtained by the preceding method.

For the maximum, assuming 14^h as the time from which T is reckoned, we get from (40)

$$B_1=-.27+.415=.145,\ B_2=-.83$$

With these values (41) gives, the first two terms only having any sensible effect in this case,

$$T=-\frac{B_1}{B_2}=\frac{.145}{.83}=0.175^h=10.5^m$$

Hence the time of maximum is 14^h 10.5^m, very nearly the same as 14^h 11^m obtained by the preceding method.

In a similar way the times T' of the mean value of y in (2) may be obtained from (43), using (40) in obtaining the constants. The value y_0, from the hour of which T' is reckoned, should be the one which is the nearest the mean value, either preceding or following.

280. From the computed values of a_c in the preceding table we get the combinations in the following tables for obtaining with a small correction applied, the diurnal mean from only two or more observations:

Combination.	Correction to be added.		
	Winter.	Summer.	Year.
	°	°	°
$\frac{1}{3}$ ($7^h+15^h+23^h$)	−0.07	+0.23	+0.10
$\frac{1}{3}$ ($7^h+14^h+21^h$)	−0.59	−0.56	−0.60
$\frac{1}{2}$ (10^h+22^h)	+0.68	−0.05	+0.21
$\frac{1}{2}$ (max. + min)	−1.15	−0.46	−0.79
$\frac{1}{3}$ ($6^h+14^h+22^h$)	−0.38	+0.62	+0.18
$\frac{1}{4}$ ($7+14^h+21_h$ *bis*)	−0.28	+0.14	−0.08
$\frac{1}{4}$ ($3^h+9_h+15^h+21^h$)	+0.02	−0.19	−0.13

For spring and fall the corrections for the year can be used without any sensible error. Approximate monthly corrections may be deduced from these by a rude interpolation which would be but little in error. Of these combinations the first is preferable on account of the corrections being both small and not varying much during the year. The sixth one is very nearly as good. The second one is important on account of its comprising the hours of observation adopted by the Smithsonian Institution. The correction required is somewhat large, but it is very nearly the same throughout the year.

It is probable that these combinations could be used without incurring much error, for all places within a long distance in any direction from Washington, and that with similar corrections obtained in the same way for a very few important places over the United States, the corrections to be used for all intermediate places could be deduced from them with considerable accuracy.

281. The following table contains the monthly averages of the temperature observations made at the Signal Service Office, Washington, during the years 1871–'82, inclusive, twelve years:

Months.	Monthly averages or values of a_s.	Differences used in the formulæ of § 269.		Computed a_s.	Residuals $O-C$.
	°			°	°
January	a_0, 33.52	(a_0-a_6), 44.72	(a_0-a_3), 19.30	33.74	−0.22
February	a_1, 36.00	(1— 7), 39.00	(1— 4), 28.43	35.90	+0.10
March	a_2, 42.34	(2— 8), 25.47	(2— 5), 31.31	42.27	+0.07
April	a_3, 52.82	(3— 9), 5.09	(3— 6), 25.42	52.96	−0.14
May	a_4, 64.43	(4—10), 20.73	(4— 7), 10.57	64.19	+0.24
June	a_5, 73.65	(5—11), 38.12	(5— 8), 5.84	73.78	−0.13
July	a_6, 78.24	(6— 0), 44.72	(6— 9), 20.33	78.10	+0.14
August	a_7, 75.00	(7— 1), 39.00	(7—10), 31.30	74.96	+0.04
September	a_8, 67.81	(8— 2), 25.47	(8—11), 32.28	67.99	−0.19
October	a_9, 57.91	(9— 3), 5.09	(9— 0), 24.39	57.00	+0.31
November	a_{10}, 43.70	(10— 4), 20.73	(10— 1), 7.70	44.07	−0.37
December	a_{11}, 35.53	(11— 5), 38.12	(11— 2), 6.81	35.16	+0.37

The differences used in the formulæ of § 269 are given in part in this table, and the others can be arranged in the same manner. With these the constants M_s and N_s are readily obtained by these formulæ, and then with these (15) gives the amplitudes and epochs of the several compounds. They are as follows:

$$
\begin{array}{ll}
A_0=55.05^\circ & \\
A_1=22.75^\circ & \epsilon_1=184^\circ\ 8' \\
A_2=\ 0.68^\circ & \epsilon_2=\ 60^\circ\ 0' \\
A_3=\ 0.85^\circ & \epsilon_3=\ 70^\circ\ 21' \\
A_4=\ 0.65^\circ & \epsilon_4=358^\circ\ 31'
\end{array}
$$

Since a_0 above corresponds to the middle of January, this is the era from which the preceding epochs must be reckoned. If reckoned from the beginning of the year, they must be increased by $si_1 \times 15.5$, the change of the angles during the half of the month.

The values of the reciprocal of k_s to be used in (15) in this case are those in the table of § 265 corresponding to $(\theta''-\theta')=30^\circ$, or those under the headings respectively of 30°, 60°, 90°, and 120°.

With the preceding values of A_s and ϵ_s (2) gives the normal temperature of any day in the year so far as it is determined by this series of twelve years of observations. The value of the angle i_1t to be used for any given day can be obtained from Table XI, Appendix, where the epochs are reduced so as to be reckoned from the beginning of the year.

With the values of M_s and N_s, using cosines and sines of the angle, or with the preceding amplitudes multiplied into k_s, and the epochs, the values of a_s are computed and given in the table above, and also the residuals in a comparison with observation. If the uncorrected amplitudes were used, we should get the true normal temperature for the middle of each month and not the average of each month, and hence it would not be comparable with the observed value of a_e. In these computations, as in making out the equation of condition to obtain the constants, each month was supposed to correspond to 30° of the angle, which is not strictly correct, but the error arising from this supposition has been found to be extremely small. The angles used should be those obtained from Table XI for the middle of each month.

With the residuals above we get from (26)

$$e=0.6745\sqrt{\frac{.5846}{12-9}}=.2981$$

the number of observations being 12, and the number of constants determined 9.

With this value of e (27) gives $r=0^\circ.086$ as the probable error of A_0, and (28) gives $r=0^\circ.122$ as the probable error of the amplitudes of the components.

From (29) we also get

$$r'=\frac{0.122}{22.75}\times 57^\circ.3=0^\circ.30$$

as the probable error of the epoch ϵ_1 of the first and principal component. Of course the probable errors of the smaller components are comparatively very great since they are inversely as the amplitudes.

These determinations of probable errors are based upon the assumption that the four components taken into account are all the real sensible components, and that the residuals are entirely abnormal and accidental. This does not seem, from an inspection of them, to be quite the case, but the indications are that any remaining real components must be very small. If more components had been taken into account there would have been scarcely any basis left for the determination of probable errors without more observed values of a_e within the period. There is some uncertainty, therefore, with regard to these probable errors, but they give us at least some idea of their magnitude.

There is no convergency in the last three of the amplitudes, which would indicate that at least the last two merely represented uneliminated abnormal irregularities if they were of the same order as the probable errors, but being considerably greater they must belong to real terms.

By means of (32) and (33), as in the case of the diurnal inequality, the times of maxima and minima can be determined, using the preceding values of the amplitudes and epochs. The value of β by (32) is found to be 13° 40′ for the minimum and −3° 58′ for the maximum. From (33) we therefore get, putting $n=-1$ in this case,

$$T=\frac{184^\circ\ 8'-13^\circ\ 40'-180}{.986}=-9.63 \text{ days}$$

for the time of minimum reckoned from the middle of January. This makes the minimum fall about the end of the 6th day of January.

In like manner we get

$$T=\frac{184^\circ\ 8'+3^\circ\ 58'}{.986}=190.4 \text{ days}$$

for the time of the maximum from the middle of January, or 206 days very nearly from the beginning of the year. This, by Table XI, makes the maximum fall at the end of the 25th day of July.

Thermometry.

282. Bodies expand with increase of temperature. The rates of their expansion therefore become measures, more or less perfect, of temperature. Temperatures are usually measured by the rate of expansion of mercury, or rather of the relative rate between it and the glass inclosure which contains it.

If a body is immersed in, or entirely surrounded by, another body, and is neither gaining nor losing heat either by conduction or radiation, it is said to have the same temperature. If the mercurial thermometer is surrounded by melting snow or scraped ice and allowed to remain until no further contraction or expansion takes place, it is said to have the temperature of freezing, and this fundamental point of the temperature is made the zero of the centigrade scale. Again, if the thermometer is surrounded by the escaping steam of boiling water within an inclosure, its temperature is said to be that of the boiling point, and this is another fundamental point, which is marked 100° on the centigrade scale.

In the expansions and contractions which arise from changing temperatures, glass does not return at once to its original state after having been heated far above the freezing point and then restored to this temperature, so that if the freezing point is determined immediately after having been heated up to the boiling point, it is found to be with different kinds of glass from 0.1° to 0.5° lower than it was before having been heated to a high temperature. This is called the *depressed* freezing point. This point, however, subsequently rises, at first rapidly and afterwards more slowly, until, in a month or less, it arrives at the state of long repose.

The distance between the boiling and the depressed freezing point is called the fundamental distance. The depressed freezing point is used now in preference to the other, because in the fundamental distance thus taken it is found from observation that there is no variation with time within the limits of the errors of observation. It is this distance in the tube of the thermometer which measures 100°, and if the bore of the tube could be made of uniform diameter in all parts, each of the equal intervals of the fundamental distance would measure one degree of temperature. But as the bore cannot be made uniform, the graduation must be such as to make each division represent equal volumes of mercury, corresponding to equal expansions. This process is called *calibration.* Where this is not accurately done, a correction must be applied, called the *correction of calibration.*

Since the depressed freezing point is used as the zero of the scale the difference between this and the freezing point of the thermometer at the time of observation, after having been subject to a gradual change with time, should be known, and a correction, called the *correction of the freezing point*, be applied. Strictly, this correction should be determined after each observation, especially for high temperatures, but for usual ranges of temperature the corresponding variations of the freezing point may be neglected, and simply a constant be applied, after the thermometer has arrived at its state of repose. This constant, however, should be determined occasionally, after not very long intervening intervals of time.

As water boils at different temperatures under different pressures, of course it must be understood that the boiling point used in the fundamental distance is that determined with the standard pressure which has been already defined in § 16. This is the one recently adopted by the International Committee of Weights and Measures, though the one used by Regnault was that of a barometric pressure of 760^{mm} at Paris, latitude 48° 50′ and altitude 60 meters. The difference, however, in the two scales is extremely small.

If the fundamental distance in any thermometer through any imperfection in its construction, or through any slight changes in the nature of the glass after construction, should be found to be erroneous, the readings of the thermometer must be corrected for this error in proportion to the temperature read.

A thermometer of which the calibration has been studied and the freezing point and boiling point determined is called a *normal* thermometer. Putting

f=the erroneous fundamental distance of a thermometer;
T=the temperature reading;
Δ_0=the correction of the freezing point;
Δ_T=the calibration correction;
Σ=the sum of all the corrections;

we then have for the corrections of a normal thermometer[2],

$$\Sigma=\Delta_0+\Delta_T+\frac{100-f}{f}T$$

283. The indications of the mercurial thermometer are not strictly the true absolute temperatures indicated by the expansions and contractions in volume of a perfect gas under equal pressures, or by the changes of pressure with equal volumes, and which, when used in the measurements of heat, make such measurements proportional to the work done. Even air does not quite fulfill this condition, since in the relations between pressure, volume, and temperature it does not entirely conform to the laws of Boyle and Charles. Even the air thermom-

eter itself, therefore, requires a small correction to reduce its indications to the true temperature, but the true amount of this reduction is somewhat uncertain, and so small that it may be neglected, except in refined physical researches.

The differences between the mercurial and the air thermometer are attributable to the variable rates of expansion of mercury and glass at different temperatures, the expansibility increasing with increase of temperature. A formula for the reduction of mercurial temperatures to air temperatures has been deduced by Prof. T. Russell, substantially in the following manner: Let

V=the volume of the bulb and tube, up to the zero-mark, of a mercurial thermometer at temperature 0°;

v=the volume of the tube from 0° to 100°, and at the temperature also of 0°;

T=the temperature indicated by the mercurial thermometer, corrected for calibration, freezing point, and erroneous fundamental distance;

τ=the true temperature;

β_s=the cubical coefficients of expansion of mercury for the several powers of τ;

γ_s=the same for glass.

We then get for any temperature τ by equating the difference of the expansions of the mercury and of the glass with the volume of mercury above the zero-mark at temperature τ,

$$V\,(1+\gamma_1\tau+\gamma_2\tau^2+\gamma_3\tau^3)-V(1+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3)=v(1+\beta_1\tau)\,\frac{T}{100}$$

neglecting the terms in the last member depending upon the second and third powers of τ as having no sensible effect.

For τ equal to 100° this becomes

$$V(1+100\beta_1+100^2\beta_2+100^3\beta_3)-V(1+100\gamma_1+100^2\gamma_2+100^3\gamma_3=v(1+100\beta_1)$$

Dividing each of the foregoing equations by V and eliminating the ratio $v : V$, by substituting in one of the equations its value found from the other we get, by transposing and reducing,

$$T=\tau\,.\,\frac{1+100\beta_1}{1+\tau\beta_1}\cdot\frac{1+\tau x+\tau^2 y}{1+100x+100^2 y}$$

in which

$$x=\frac{\gamma_2-\beta_2}{\gamma_1-\beta_1}\qquad\qquad y=\frac{\gamma_3-\beta_3}{\gamma_1-\beta_1}$$

Since each of the factors in the preceding expression by which τ is multiplied is nearly equal to unity, this expression may be put into the following more simple form:

$$T=\tau[1+\beta_1(100-\tau)]\;\;[1+x(\tau-100)+y(\tau^2-100^2)]$$

If the expansions of glass and mercury were strictly proportional to the temperature, the coefficients of the second and third powers of the temperature would be zero, and then the last factor in the preceding expression would vanish.

With this part of the expression, putting the coefficient of glass β_1 equal to 0.000026, we get a plus correction at the temperature of 50° C. of 0.065°, and at the temperature of —40° C., a correction of —0.145° C. These corrections arise from the varying capacity of the thermometer bore at different temperatures. It is sometimes called the Poggendorf correction, from the name of the physicist who first pointed it out.

If the coefficients of expansion of mercury and of glass were known in all cases, the values of $(T-\tau)$, that is, the correction to reduce the mercurial temperatures to air temperatures, could be readily obtained. But these are not generally known with sufficient accuracy, and therefore Professor Russell has determined the values of x and y for several thermometers from observation by comparing the mercurial with an air thermometer in different parts of the scale and thus making out a number of equations of condition from the average results of many comparisons, and solving them by the method of least squares. With the values of the unknown constants thus obtained, he computed the following differences $(T-\tau)$ between air-thermometers and mercurial thermometers of different kinds of glass. Some of his results are given in degrees Fahrenheit in the following table to show the usual amount of such corrections and their variations with different kinds of glass used in the construction of thermometers:

Temperature.	B. 9707.	B. 9705.	G. 4470.	G. 7375.	G. 7376.
° F.	° F.	° F.	° F.	° F.	° F.
− 38	−0.49	−0.43	....	....	....
− 28	−0.39	−0.35	....	+0.23	+0.26
− 18	−0.31	−0.28	....	+0.15	+0.20
− 8	−0.24	−0.22	....	0.00	+0.11
+ 2	−0.17	−0.15	....	+0.03	+0.06
12	−0.11	−0.10	....	−0.05	+0.02
22	−0.05	−0.05	....	+0.08	−0.05
32	0.00	−0.00	0.00	0.00	0.00
42	+0.16	+0.18	+0.07	+0.02	+0.05
52	+0.20	+0.13	+0.16	+0.09	+0.10
62	+0.27	+0.20	+0.18	+0.06	+0.06
72	+0.29	+0.23	+0.18	+0.05	+0.05
82	+0.29	+0.22	+0.16	+0.03	+0.02
92	+0.33	+0.27	+0.14	+0.03	−0.01
102	+0.27	+0.22	+0.14	−0.02	−0.05
112	+0.30	+0.21	+0.14	−0.02	−0.06
+122	+0.24	+0.22	+0.12	−0.08	−0.08

The first two, Baudin, were made of crystal glass; Green, 4470, of Corning glass; and the last two of a new kind of glass tried by Green.

Thermometer exposure.

284. Such a position of the thermometer that its temperature will at all times be the same as the temperature of the air is what is sought in thermometer exposure. Such a position, however, it is very difficult to find. We have seen § 92, that the temperatures of bodies depend upon

a great many circumstances, upon local surroundings, size, shape, and constitution, so that different bodies with the same local surroundings and the same bodies removed a short distance only where the surroundings differ but little, may have very different temperatures. But the temperature of the air, which we wish to determine, does not depend upon these local and other conditions, but upon the conditions to which it has been subject in passing over the earth's surface for some time before arriving at the place of observation. Different conditions give different temperatures, and therefore all bodies, including thermometers, have their own temperatures, differing not only from the air temperature but also from one another. It is therefore difficult to so expose a thermometer that its temperature will at all times be that of the air.

If the stratum of air in contact with the earth's surface were in perfect repose it would have nearly the same temperature both by day and by night through conduction from one to the other, but as it is being continually mixed up with the strata above, the whole mass of air, even up to a moderate altitude only, cannot follow to the full extent the diurnal changes of the earth's surface, but is, when clear, cooler during the day and warmer during the night; for since clear air has very little radiating and absorbing power, and the solar rays especially are mostly either reflected or transmitted through to the earth's surface, the diurnal changes in temperature would be small if it were not for contact with the earth's surface. It is different with the earth's surface and with bodies exposed in the atmosphere. On account of their greater absorbing powers they follow more closely the changes between day and night in the solar heat received, and consequently the amplitudes of diurnal change of temperature are greater. But this effect upon the thermometer exposed can be avoided by having it in the shade, and therefore one important condition in thermometer exposure, to obtain the air temperature, is that the direct solar rays should be excluded. The thermometer should also be protected from all reflected rays of the sun's heat, either from the atmosphere, the earth's surface, or surrounding bodies, since these have precisely the same effect in proportion to their intensity, but this of course is small in comparison with that of the direct rays of the sun.

As the temperatures of the black and bright bulb thermometers exposed in the sunshine are different, the same is, in some measure, so from the effect of the reflected solar rays, since reflection does not change that quality in the sun's rays by which the bright bulb is made to stand at a lower temperature than the black. If black and bright bulb thermometers, therefore, are exposed in most thermometer shelters, the temperatures of the former are found to be a little greater by day in clear weather, since some of the reflections from the air and surrounding objects get through the louvres by one or more reflections.[4] If the black and bright bulb thermometers show the same temperature it indicates that the thermometers are well protected from these reflections.

285. We have seen that the temperature of a body in the shade, not considering heat gained or lost by contact with the air, depends upon the temperature and the completeness of its inclosure, and that this inclosure in case of a body in the air near the earth's surface is made up of the earth's surface and all surrounding bodies which subtend a solid angle at the center, in whatever manner they may be placed, and if there is a subtending body in all directions which radiates and reflects as much heat back to the body as the body radiates in all directions, the inclosure is said to be complete. The body then, in the state of static equilibrium of temperature, has the same temperature as the inclosure if that be a uniform temperature, but if not, it will be very nearly an average temperature, especially if different parts of the inclosure do not differ very much in temperature. But it has been shown, § 86, that the clear atmosphere above and around the thermometer or other body does not form a complete inclosure, but that more heat escapes from the body in the direction of space than is returned, and hence the tendency is for the body to stand at a lower temperature at night than it would if the inclosure were complete. Another important condition, therefore, in thermometer exposure is to complete the inclosure by interposing some body, as a roof or a shelter, between the body and open space above.

During the day the earth's surface and surrounding objects become heated often much higher than the air and the reverse at night, and as these are a part of the inclosure of an exposed thermometer, the tendency is, so far as their effect goes, to raise or depress the temperature of the thermometer to the same, and this effect is proportional to the amount of solid angle subtended by these warmer bodies. The thermometer, therefore, should not be exposed over a surface the nature of which is to become abnormally heated during the day or cooled during the night, or near large bodies with so great capacity for heat that their temperatures change but little between day and night. The old rule, therefore, of exposing a thermometer on the north side of a building, facing a lawn with trees and grass sod, was a good one so far as the latter was concerned, since a grassy surface does not become abnormally heated during the day as many bare and dry surfaces with soil beneath with little thermal conductivity; but it was a very bad one so far as it related to the north wall, especially if this were a massive wall of stone or brick. For such a wall experiences but very slight diurnal changes of temperature, especially in calm weather, and as the thermometer was generally close to it, it subtended a very large solid angle, and hence, from what has been stated, its effect in depressing the temperature by day and increasing it at night is very great in clear weather when the diurnal changes are large.

286. The modern device of providing shelters of various kinds for the thermometers is only a very partial remedy of the difficulty; for the same reason that the temperature of the thermometer is too great

when exposed in sunshine or to the reflected rays of the sun, and too low at night from radiations into space, the temperature of the shelter itself is likewise too high in the sunshine and too low at night, and as this becomes an inclosure to the thermometer and the latter therefore tends to assume the same temperature, the shelter itself needs to be sheltered. And this is really done in the case of double roofs and double louvers, the outer ones being shelters for the inner ones. This is a partial remedy, since there is more matter to be heated and cooled, and this is to be done by two radiations and two absorptions, but if the periods of the diurnal changes were longer so that all the temperatures could arrive at the state of static equilibrium, the inner inclosure would assume the temperature of the outer, and the thermometer that of the inner inclosure, and there would be no advantage from the double inclosure.

287. But guarding against all the conditions which cause the extremes of temperature in the thermometers exposed, and which cause it mostly to vary from the air temperature, we still do not have an exposure which necessarily gives the air temperature, since this latter, as has been explained, depends upon entirely different circumstances. In fact the exposures would generally give only very rough approximations to the true air temperatures if it were not for the effects of the conduction and convection of the air, and these effects become geeater as the amount of surface exposed to the air is greater in proportion to the mass. It is on this account principally that advantage is found in having double louvers and double roofs. If a thermometer were exposed in air perfectly quiet, and the difference of temperature between the bulb and the air did not give rise to convective currents, the stratum of air in contact with the bulb would have the same temperature as the bulb, and there would be a gradual temperature gradient in all directions from the bulb, greater near the bulb and less further off, by which heat would be conducted away from it or into it, as the case might be; but the conduction in this case would be very small and the temperature of the thermometer bulb might be much higher during the day and lower during the night than that of the air. Where, however, there is only a little ventilation this gradient is very short and the temperature of the bulb cannot differ much from the air temperature. It has been shown from theoretical considerations, § 85, that as the ventilation is increased this difference of temperature is diminished, and that with infinite ventilation it must entirely vanish. We also know from experience that the temperature of a sling thermometer is readily brought down sensibly to that of the air, when slung in the shade, and even in the sunshine with a cylindrical bulb the difference is only about one degree Fahrenheit.

If the shelters require to be sheltered and to be ventilated in order to reduce them to the air temperature, so that the thermometer exposed within may give that temperature, and if thermometers can be so readily reduced directly to that temperature, it would seem best to dispense with shelters entirely, for the thermometers can be reduced to the air temperature more conveniently than the shelters, and where shelters are not ventilated, either by the prevailing breezes or by some artificial arrangement, they necessarily give in clear weather a temperature too high by day and too low by night. In the case of maximum and minimum thermometers, however, if we wish to obtain the extremes of the true air temperature, the best that can be done is to expose them in some kind of a shelter, however much the temperatures thus obtained may differ at times from the true temperatures.

288. With regard to the locality and altitude at which air temperatures should be observed, this depends very much upon what we need and upon our object in such observations. If we wish to obtain a sort of general temperature of the air of any place which is independent of local conditions which may make the temperatures differ much in different places even in the same vicinity, and one which is not affected by the great extremes of day and night which are often found near the earth's surface but not at a very moderate altitude above, and also one which is comparable with the surrounding temperatures and temperatures generally over the earth, the exposures should be at a considerable altitude, where there is a free atmospheric circulation and few absolute calms and local surroundings which cause variations of temperature. For such a purpose temperatures observed on the tops of buildings not differing much in height are most suitable, provided they are taken with a sling thermometer, so that they may not be vitiated by reflections from roofs and surrounding buildings, as they are in the case of shelters where there is little or no wind. The thermometer, of course, must be slung in the shade, but the effects of the reflected heat from the atmosphere, from roofs, &c., are entirely overcome by the ventilation in this method.

It must be remembered, however, that temperatures in and over a city are in general higher than those in the surrounding country and are therefore not strictly comparable. This is especially so with regard to the extremes of the seasons and of day and night, but very much less so in the annual and diurnal means.

If a station is desired which will give the greatest extremes of temperature of both day and night, then one should be selected in some low valley with a dry, sandy soil, for on account of its small thermal conductivity such a soil, for well-known reasons, becomes very highly heated during a clear day, and its temperature is reduced very low at night, and the temperature of the stratum of air resting upon it, in calm weather, follows closely after it, while above at a very moderate altitude the diurnal extremes are comparatively very small. The gradient of temperature decreasing with increase of altitude becomes very great near the earth's surface by day, when clear, but at night it not only becomes very small but is generally completely reversed, and the air is colder at the earth's surface than at some small altitude above. The observations of different stations in such cases are, therefore, not at all comparable unless they are made at the same altitude above the earth's surface.

Where the temperature is observed four feet above a grass sod, it is very nearly that of the lower part of the atmosphere in general, especially where there is a free circulation, for the dampness of the soil and the consequent evaporation always going on, as well as conduction of heat into the interior of the earth, keeps the surface and lower stratum of air from becoming very much heated, but still the air at such a station is subject to the extremely low nocturnal temperatures which frequently take place. But as these occur only very near the earth's surface they have too much influence in deducing, from the observations of such a station, the general temperature of the atmosphere.

The greatest extremes of low temperature in such cases are given where the stations are near the earth's surface in valleys surrounded by bare hills or mountain sides. During clear nights these bare surfaces are cooled down very low by radiation into space, and the cold air in contact flows down into the valleys and over the plains below, covering the earth's surface with a thin stratum of air very much colder than that only a very little distance above. At some place only a short distance away, but a little more elevated, these low extremes of temperature would not be observed, so that not only the extremes but the mean temperature of the day would be much less, and all this would be due to the different effects of very local circumstances. The results obtained from such stations may be interesting and useful, but they are not such as should be used in determining the general air temperature of any region or country.

Reduction of temperature to sea-level.

289. Since temperatures decrease with increase of altitude, they are not comparable, when observed at different altitudes, with those of surrounding stations, in studying the general effect of differences of latitude and longitude on temperature, and in laying down isotherms on charts which are intended to exclude the effects of altitude. They require to be first reduced to some general level, usually that of sea-level. This reduction, however, is rather a vague and uncertain one. In order to have a temperature comparable with other stations at sea-level, it is necessary to have the temperature which would exist vertically under at sea-level, if the mountain or plateau were not there, instead of the one observed above. But from the observed one we have no means of determining what the required one is, since not only the observed temperature may differ very much from the temperature that would be found in the open air at that altitude, but also the exact rate of increase of temperature in descending from the upper station through the free air to the earth's surface is unknown. This has been determined in

some parts of the earth by comparing temperatures observed at some low station with those observed on the top of some high, and generally distant, station. But the observations at both stations are those of surface temperatures, which are subject to all the local influences and large fluctuations, annual, diurnal, and abnormal, of surface temperatures, and do not correspond with those at the same altitudes in the open air. The annual and monthly rates thus obtained for a few places on the earth are given in the table of § 140. From these, it is seen, the mean annual rate differs considerably in different places, and also that there is an annual inequality in this rate which has its maximum late in the spring or in the first part of summer. The summer rates, upon the whole, agree pretty well with those obtained from balloon ascents in the summer season in the open air, which up to the altitude of one mile give an average of $0^{\circ}.68$C. At higher altitudes, however, the rate is very much less.

In the case of mountain peaks and sharp ranges, rates of reduction to the level of surrounding plains, or even to sea-level, might be obtained from observation for each month of the year which would give satisfactory reductions for monthly averages, but if these were applied to individual cases of reduction, affected by the large diurnal and abnormal fluctuations of temperature, the results would be found to be very unsatisfactory; for in such reductions it would be necessary to know the rate of change of temperature with increase or decrease of altitude, as affected by these diurnal and abnormal fluctuations.

The greatest uncertainty in reductions to sea-level is in the case of extensive and high plateaus. On these the temperatures, even a mile or more above sea-level, are often found to be not very much less than on the plains in the same latitude near sea-level. Any of the ordinary rules and rates of reduction, therefore, applied to these temperatures, give a temperature generally which is much too high to correspond with temperatures on each side in the same latitude at sea-level.

In preparing isothermal charts in which it is desired to exclude the effects of altitude, some kind of reduction is necessary, but in this case the uncertainties of reduction caused by diurnal and abnormal fluctuations do not come in, since they are eliminated from the averages, and if the observations of the higher stations are excluded, which is generally best, rates of reduction can generally be obtained which will be satisfactory for the stations of no very great altitude. This is a matter, however, for which there are no certain rules or data, and in which much depends merely upon a good judgment. It is not worth while to attempt any nice refinements in reduction, and the formula used should be a simple one, so that the temperatures of the charts can be easily reduced back again to the altitudes of observation. This should especially be so where reductions are made in the case of temperatures on large and elevated plateaus.

Abnormal variability of diurnal temperature.

290. Besides the annual and diurnal variations of temperature obtained by the harmonic analysis, there are abnormal variations depending upon various causes, having no regular amplitudes and periods, which are in no way connected with, or dependent upon, the others. This variability differs in different places and at different seasons at the same places, and it is interesting in a climatic point of view to have some means of comparing this variability for different places and seasons of the year.

A measure of this variability is obtained by taking the differences of mean diurnal temperature from day to day for each month, or any other intervals, and taking the average without regard to sign. The greater and the more frequent and sudden are the abnornal changes of temperature the greater is this average, so that it may be regarded as a measure of the variability of climate so far as the variations depend upon abnormal causes. In this process the normal annual change of temperature is so small each day that its effect upon the differences can be regarded as insensible. In this way Dr. Hann[5] has determined the mean monthly variability of the diurnal temperature for 90 places on different parts of the globe, on both continents and in both hemispheres, from series of observations from one to ten years. The variability was found to increase in general from the sea-coasts toward the interior of the continents, and also to increase a little with increase of altitude. This latter result was hardly to be expected, inasmuch as the annual and diurnal variations diminish with increase of altitude. In general the variability was found to be considerably greater in winter than in summer.

The following table contains the results in centigrade degrees of a few places in North America, on the sea-coast and in the interior, with the number of years' observations used:

Table.

Place.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.	Year.
Sitka (3)	2.1	2.0	1.5	0.9	0.9	0.9	0.8	0.7	1.1	1.5	1.6	1.7	1.3
San Francisco (9)	1.4	1.5	1.2	1.3	1.2	1.0	0.7	0.8	1.2	1.6	1.4	1.4	1.2
New Orleans (2)	3.6	3.1	2.6	2.4	1.5	1.7	1.1	1.1	0.8	2.2	2.4	2.9	2.1
Providence (6)	3.6	3.4	2.8	2.7	2.6	2.1	1.8	2.0	2.4	3.2	3.2	3.7	2.8
Washington (10)	3.7	3.7	3.3	2.7	2.0	1.4	1.1	1.2	1.6	2.6	3.5	3.6	2.5
Saint Louis (6½)	3.8	4.1	4.2	3.5	2.6	2.2	1.9	1.8	2.5	2.6	3.7	4.2	3.1
Laramie (5)	5.1	4.8	4.4	3.9	3.1	3.0	2.9	3.4	3.6	4.1	4.6	5.5	4.0
Red River and Winnipeg (7)	6.2	4.9	4.1	3.3	3.6	2.9	2.7	2.6	2.9	3.2	4.0	4.6	3.8

Actinometry.

291. There are two general methods of measuring the heat received from the sun, the *dynamic* and the *static*. As defined by Sir John Her-

schel,[e] the former "consists in ascertaining the amount of physical change of a nature susceptible of being measured, effected upon some object by a given sectional area of sunbeam in a given time, such, for instance, as the dilatation of a liquid, the melting of ice, or the raising of a given quantity of water a number of degrees;" and the latter, "in equilibrating the heating power of sunshine on some body, as a blackened thermometer bulb, with the cooling influence of the radiation of the same body, the rate of cooling and of receiving and radiating heat being supposed to be proportional to the change of temperature."

The first attempt at measuring the intensity of the sun's thermal radiation by the dynamic method was made by Herschel in 1824. "A glass vessel full of inked water was exposed alternately five minutes in sunshine and in shade, the change of temperature being noted by a very delicate thermometer immersed in the liquid, and the solar effect per minute measured by the difference of the minute changes observed to take place in sunshine and in shade."[76] A similar method was employed in the experiments at the Cape of Good Hope (1836–'37), the sun, nearly vertical, being allowed to shine directly on a cylindrical vessel of water. From these observations it resulted that the direct heating effect of a vertical sun at sea-level is such as would suffice to melt 0.00754 inches in thickness per minute from a sheet of ice exposed perpendicularly to its rays.

Subsequently Herschel invented an actinometer, called *Herschel's actinometer*, which, as described by himself,[f]—

> Consists of a blue liquid (ammonio-sulphate of copper), inclosed in a glass cylinder, one end of which is closed by a screw working in a tight collar, to admit of a small change of capacity when the liquid becomes too much dilated by heat, while the other is soldered on to a thermometer tube, by which the liquid measures its own dilatation, the cylindrical portion acting as the bulb of a thermometer. The actual temperature of the liquid (on which the dilatability depends) is ascertained by an interior thermometer occupying the axis of the cylinder, and whose stem penetrates the axis of the adjusting screw, and is read along its exterior prolongation. The instrument being several times alternately exposed for one minute in the sun and shade, and the changes of volume in each case read off on the scale, the differences or sums of the mean changes, according as the action has been in the same or the contrary direction, gives the dilatation produced by the sunshine alone (freed from the disturbing influences) corresponding to the actual temperature of the liquid, which, being reduced by an appropriate table to give the temperature acquired, affords a measure of the effect of a given sectional area of sunbeam in heating a definite volume of liquid.

If we put

T=the increase of temperature during one minute due to the action of the sun's rays;

τ =the observed increase during the minute;

τ_1=the observed increase during the preceding minute;

τ_2=that of the succeeding minute,

we shall then have

$$T=\tau-\frac{\tau_1+\tau_2}{2}$$

In this it is assumed that the rate of increase or decrease of temperature from other causes than the direct action of the sun's rays is uniform during the three successive observations; or, in other words, that second differences may be neglected.

The following is an example, taken from the observations made by Kaemtz on the summit of the Faulhorn, 1832, showing the manner in which such observations are made:

Exposure.	Hour.	Actinometer.	Difference.	Radiation.
	h. m.	°	°	°
Shade..........	7 26	26.2		
Sunshine	7 27	30.2	+ 4.0	....
Shade..........	7 28	51.6	+21.4	16.4
Sunshine	7 29	57.5	+ 5.9	....
Shade..........	7 30	80.3	+22.8	16.9
Sunshine	7 31	86.2	+ 5.9	....

From these data we get, by the preceding formula,

$$T=21.4-\frac{4.0+5.9}{2}=16.4$$

for the sun's radiation during the second minute, and in the same manner that of the fourth minute is obtained.

In these observations the temperature of the liquid was considerably below that of the air at the commencement, and hence there was an increase of temperature, even while the instrument was in the shade, and hence the differences are all positive. Otherwise they might have been, and generally are, alternately plus and minus.

At the time of the last observation given above the liquid in the instrument had expanded so much that a new adjustment by means of the silver screw at the bottom of the cylinder became necessary before the observations could be continued.

With the change of temperature during one minute, due to the sun's rays, and freed from the disturbing influences, and the known thermal capacity of the vessel and liquid heated, the absolute amount of heat received from the sun during the minute becomes known.

292. The results obtained with this actinometer are now known to be too small. This, however, is the fault mostly, if not entirely, of the method of using it and not of the actinometer itself. In order to have a true measure of the sun's thermal intensity the instrument should give changes of temperature proportional to the time when there is no sensible change of that intensity during the time. This it does not do. At first the rate of expansion of the liquid due to the sun's heat is smaller, but gradually increases until after one minute or more it acquires the regular rate, which is not necessarily that of uniformity, but usually a gradually decreasing rate, since the higher the temperature

of the vessel and the liquid the more it cools from radiation and conduction of the air. This was shown by Kaemtz,[7] who made the following experiments upon the rate of heating and cooling by observing every tenth second instead of each minute only:

Time.	Actinometer in sun.	Change in 10 seconds.	Actinometer in shade.	Change in 10 seconds.
″	°	°	°	°
0	22.71		66.49	
10	24.97	+2.26	65.66	—0.83
20	28.49	3.52	63.77	1.89
30	32.04	3.55	61.58	2.19
40	35.80	3.76	59.29	2.29
50	39.68	3.88	56.83	2.46
60	42.39	3.71	54.58	2.25
70	47.28	3.84	52.21	2.37
80	51.21	3.98	49.78	2.48
90	55.20	3.99	47.39	2.41
100	59.02	3.82	44.94	2.45
110	62.98	3.96	42.51	2.43
120	66.40	3.51	40.14	2.37

It is seen from this table that the rate of heating was the least during the first ten seconds and gradually increased for more than one minute instead of being the greatest during the first ten seconds and gradually decreasing, as it should, according to the regular law. For as the vessel and liquid become heated up they lose heat faster by radiation and conduction, and therefore the rate should gradually diminish from the start as it is found to do at the end of the two minutes. The effect of the sun's heat during the first ten seconds was mostly upon the glass vessel and liquid in contact, and only in a small measure upon the part in the interior until there is formed a temperature gradient from the surface of the glass to the central part of the liquid such that the heat is conducted in and raises the temperature of the whole mass throughout in proportion to the heat received. Before this the glass receives more than a proportional part of the heat, and as this is not indicated by the instrument, it does not furnish, during this time, a true measure of the heat received.

After the sun's rays are cut off by the screen and the glass and liquid begin to cool, the first effect is to cool the glass more than the rest of the cooling mass until the temperature gradient becomes reversed, and great enough to convey the heat away equally from all parts of the mass. Only after this takes place are the indications of the instrument proportional to, and a true measure of, the loss of heat. This, according to the last column in the preceding table, did not take place in Kaemtz's experiments until near the end of the second minute, for the normal rate of cooling must be a decreasing one. It is readily seen from these experiments, that where there is a regular alternating of sunshine and shade every minute, the regular normal rates of heating and cooling are not obtained, but simply the initial smaller rates in which the effects are more upon the glass than upon the liquid, and consequently the differences, and the solar radiation obtained from them by the preceding formula, are too small.

293. Pouillet's actinometer (called by himself *pyroheleometer*) consists of a cylindrical vase one decimeter in diameter, and 14 or 15 millimeters high, made of silver or plated metal, very thin, and contains about 100 grams of water. There is a thermometer in the vase held by a cork fastened to a tube of metal sustained by two collets in which it swings freely, so that the whole apparatus can be turned around the axes of the thermometer to agitate the water in the vase and render the temperature of the water uniform through the whole mass. The upper surface of the vase which receives the solar rays is carefully blackened with lamp-black, and it is maintained perpendicular to the rays by arranging the apparatus so that the shadow of the vase shall always cover a circle mounted perpendicular to the axis at the other extremity of the tube. The experiments are made in the same manner as with Herschel's apparatus, except that the alternate intervals of sunshine and shade are 5 minutes instead of one minute; and the final results are obtained in the same way from the differences of the temperatures observed at the end of each interval.

In this apparatus there is a similar retardation in arriving at the normal rate with which the liquid receives and loses heat after change from sunshine to shade, or the reverse, as in the case of Herschel's, and consequently it has precisely the same defect, but perhaps not quite to so great a degree, since the intervals are longer, and consequently the rates of heating and cooling are the normal ones during at least a part of the interval.

Notwithstanding the constant agitation of the water in the vase, the radiating face of the apparatus is found to stand at a higher temperature than the water. The agitation is not sufficient to detach a stratum of water adhering to the inside of the face, and this forms a stratum of not very great thermal conductivity. The changes of temperature, consequently, in the face receiving the sun's rays, and of the stratum of water adhering to it, are much greater in the heating and cooling than those indicated by the thermometer in the central part of the water.

294. *Crova's actinometer*,[8] in which the liquid is of alcohol, although of a form differing from that of Herschel's or Pouillet's, is upon the same principle, and the observations are made and treated in the same manner, except that the readings of temperature are not made for some time after the first exposure and reversals from sunshine to shade and the reverse, when the normal rates of heating and cooling are established, which in the case of Herschel's actinometer is more than one minute, and in others a greater or less time, to be determined in each apparatus by experiment. By proceeding in this manner with his appa-

ratus he seems to have obtained consistent and satisfactory results, from which he has deduced a value of the solar constant considerably greater than that usually obtained with Herschel's or Pouillet's actinometer with observations made in the manner described in § 291.

With observations, however, made and used by Crova's method, there is no reason why the other instruments should not give results equally as good and satisfactory.

295. In the static method, when the temperature of a body is raised so high that the rate of losing heat by radiation or otherwise is exactly equal to the rate with which it receives heat from the sun, it of course becomes stationary. But the greater the temperature the greater is the rate of losing heat, and so, when the temperature becomes stationary, the greater is the rate with which the body is receiving heat. Hence the height of this stationary temperature above the temperature at which the body would stand in case it received no heat from the sun, which is usually assumed to be the shade temperature of the body, becomes a relative measure of the intensity of solar radiation.

Newton was the first to determine—in a crude manner—the strength of solar radiation by this method. He observed the temperature of a sandy soil with little thermal conductivity, when exposed to the sun's rays, and found for the latitude of London an excess of 65° F. above the shade temperature. This was regarded as a measure, though imperfect one of course, of the strength of the solar radiation at the time. Such observations made at other times, under precisely the same circumstances, would have furnished rough relative measures of the solar radiation, but they would not have been comparable with those made on different soils and under other circumstances.

Similar observations were made long afterward by Lambert,[9] Daniell,[10] Sabine, and Perry. Leslie[7] also applied his differential thermometer upon the same principle. He made a series of observations at Edinburg with that instrument, of which one bulb was blackened and the other bright and both exposed in the open air, the differences of the temperatures indicated by the two being taken as a measure of the intensity of solar radiation.

The temperatures observed in all these cases were those of bodies in the open air; and the cooling effects of radiation, conduction, convection, &c., were dependent upon so many local circumstances, that the excess of temperatures observed above the shade or air temperature could not be regarded as even a rough relative measure merely of the rate of solar radiation. This excess, when the thermometers were placed in calm air, was very much greater than when they were ventilated by even a very gentle breeze only, so that it changed with the changes of every breeze, and the observations made at different times and places were not comparable.

A great improvement upon these first rude attempts at measuring the intensity of solar radiation by the static method, first suggested by Sir John Herschel, consisted in putting the black-bulb thermometer in a vacuum within a glass envelope. With this arrangement the cooling depends upon radiation alone and is not affected by the conduction of the air and varying currents. The excess of temperature over the air temperature, especially if this is the temperature of the envelope, as it generally is very nearly, furnishes a much better relative measure of the rate with which heat is received from solar radiation.

At the meteorological observatory of Montsouris an apparatus is used, called the *Arago-Davy actinometer*, formed by Marie-Davy by reconstructing one found amongst the collections of Arago, which consists of two thermometers *in vacuo* with glass envelopes, the one with a blackened and the other a bright bulb. Where both are exposed, side by side, to the sun's rays at the height of about four feet above a grass sod, far away from any shelter from the sun's rays, the difference between the indications of the two thermometers is assumed to be a relative measure of the intensity of solar radiation at the time.

296. None of these static methods gives an *absolute* measure in heat units of the rate with which heat is received, or even a true *relative* measure. A true measure of any kind must be proportional to the thing measured, but since the absolute radiating power of a body does not increase in proportion to the increase of its temperature, increases of temperature are not proportional to the increased rates of cooling, and consequently to the increased rate of receiving heat when the temperature is stationary, and hence the differences observed are not proportional to the rate of receiving heat from the sun, and consequently not a true measure of it.

The true relation between the absolute intensity of solar radiation and the difference of temperature between the black-bulb thermometer and envelope ($\theta-\theta'$) is given by (94), § (97). By transposing we get for a spherical black-bulb *in vacuo*, putting in this case, by (87), $\rho=\frac{1}{4}$.

$$(1) \qquad I=4B\mu^{\theta'}[\mu^{\theta-\theta'}-(1-m)]$$

In the case of a complete inclosure m vanishes, but as both the glass envelope and the atmosphere around and above on the one side does not form a complete inclosure, but allow some of the radiations of even a dark body to escape through into space, and so it does not receive as much heat as it radiates when at the same temperature as the inclosure, but the equilibrium only takes place when it has a little lower temperature, m must have such a value as to satisfy (1) with the observed value of ($\theta-\theta'$) at night, when $I=0$. The value of ($\theta=\theta'$) on a clear night is about —1.5°. With this value, we get from (1) by putting $I=0$ $(1-m)=0.9886$, and hence a value differing little from unity. With this value and the value of $B=1.146$, we get from (1)

$$(2) \qquad I=4.584\mu^{\theta'}(\mu^{\theta-\theta'}-.9886)$$

for the intensity of solar radiation when the difference between the temperature of the black bulb and the glass inclosure is equal ($\theta-\theta'$)

The temperature of the air, however, cannot be used strictly for that of the inclosure unless there is considerable ventilation of the inclosure.

It is seen from (2) that $(\theta-\theta')$ is not proportional to I and therefore not a true relative measure of it, as usually assumed. For instance, with $\theta'=0^\circ$, and $(\theta-\theta')=30^\circ$ we get $I=1.24$ for the solar intensities in calories as received at the time. But with $\theta'=30^\circ$ and $(\theta-\theta')=30^\circ$ we get $I=1.56$ calories. Hence the same difference $(\theta-\theta')$ gives a considerably larger intensity in the last case than in the first, and the same differences of $(\theta-\theta')$, therefore, indicate a larger intensity in the summer than in the winter season, and in general in the equatorial than in the polar regions.

With the constants properly determined (2) gives not only a true relative measure of the intensity, but the absolute number of heat units received in a unit of time, which in the formula is one minute.

The numerical constant 4.584 is merely provisional and approximate, but as it gives a value of I which differs but little from intensities obtained by other reliable methods, it is perhaps not much in error. Its value and also that of μ would no doubt be somewhat different for a perfect vacuum.

297. Eliminating $B\mu^{\theta'}$ from the first two equations of § 99, by subtracting the latter from the former we get in case of spherical black and bright bulbs, in which case $\rho=\frac{1}{4}$,

$$I=K\mu\theta_1(\mu\theta^{\theta-\theta_,}-1) \tag{3}$$

in which, by (96), § 99,

$$K=\frac{B}{\rho-\rho_,}=\frac{4B}{1-4\rho_,}=4B(1-m)c \tag{4}$$

In this expression of K the value of c would be given by (96) if the value of $\rho_,$ were known; but for reasons given in § (93) this differs for different bright bulbs, and the data upon which its value depends are too uncertain for its accurate determination. The value of, c therefore, must be determined from observations for each pair of black and bright bulb thermometers, by means of (95), each observation of the quantities θ, $\theta_,$, and θ' in this equation giving an equation of condition from which c can be determined by the method of least squares.

By comparing (2) with (3) it is seen that $\theta_,$ in the latter takes the place of θ' in the former, and therefore the latter form of expression has the advantage of having the temperature of the bright bulb, θ' which can be easily and accurately observed, while the other contains instead the temperature of the inclosure, θ', which can only be obtained by ventilating it and observing the air temperature. In the latter expression, however, there enters the unknown constant c, to be determined, in the manner just stated, from observation. In this constant are included the uncertain value $\rho_,$, and also the value of m, in (96), § 99. The Arago-Davy actinometer, therefore, is a very convenient apparatus for obtaining data with which, by formula (3), not only the true relative, but likewise the absolute, intensity of solar radiation may be deduced.

It should be borne in mind, however, that the intensity obtained by means of this apparatus, as well as by the black-bulb thermometer *in vacuo* simply, is that of both the direct rays from the sun and of the rays reflected back from the atmosphere, that is, of the whole heat coming both directly and indirectly from the sun to the earth, while the intensity measured by the dynamic actinometers is that of the direct rays only, with generally the reflected rays from a small part of the sky in the vicinity of the sun where this is not completely cut off by the screen. The intensities, therefore, given by the static actinometers should be a little greater than those given by the others.

298. With two or more values of I, either by the dynamic or the static methods, at times suitably chosen, (101), § 50, furnishes as many equations of condition for determining the solar constant A and the diathermancy constant p. The times of observation should be so selected as to have a considerable range of altitude of the sun, or of the values of ε in the formula. The value of c in that formula being supposed to be known, two observations are sufficient to determine A and p, but it is best to have hourly or half-hourly observations from morning until evening of a very clear day, in which the state of the atmosphere may be supposed to remain very nearly the same all the day.

As (101) is not a linear expression, it is necessary, where a number of equations are formed to be solved by the method of least squares, to first assume approximate values of A and p, such that the squares and higher powers of the corrections to these assumed values may be neglected in forming linear equations for the determination of these corrections.

If in (101), § 50, we put

$$(5)\qquad I=I_1+\delta I \qquad A=A_1+\delta A \qquad p=p_1+\delta p$$

we get by development and neglecting squares and higher powers of δA and δp, and also the correction in the last very small term in (101) depending upon δp,

$$(6)\quad I=A_1(p_1^{\varepsilon}+.02\frac{\varepsilon(\varepsilon-1)}{2}p_1^{\varepsilon-2} \qquad \delta I=p_1^{\varepsilon}\,\delta A+A_1\,\varepsilon p_1^{\varepsilon-1}\delta p$$

But it may happen sometimes, where observations are made at given regular intervals during the whole or a greater part of the day, that there is a gradual change in p, and such that the change may be regarded as being proportional to the time, so that the whole of the observations, on that account, cannot be represented by (101). We should, therefore, have not only the correction due to the assumed value of p, but likewise that due to the gradual change in the diathermancy of the atmosphere. Instead of δp above, therefore, we should have to introduce $\delta p + t\delta' p$, that is, an additional term in the correction of p proportional to the time t. The preceding expression (6_2) would then become

$$\delta I = p_1{}^{\epsilon}\,\delta A + A_1 \epsilon p^{\epsilon-1} \delta p + A_1 \epsilon p_1{}^{\epsilon-1} \tau \delta' p \tag{7}$$

The values of I_1 being computed by (6_1) which satisfy the assumed values of A_1 and p_1, with the residuals, or differences between these and the observed values of I, that is, with the values of δI, are formed from (7) as many equations of condition for determining the corrections δA and δp of the assumed values of A and p, and also $\delta' p$. At least three equations are necessary for this purpose, but it is best to have more and to solve by the method of least squares.

Where there are no apparent differences between the observed values of I for equal altitudes of the sun during the forenoon and afternoon, or where a regular series of hourly or half-hourly observations are made through the day, so that forenoon and afternoon observations can be combined, thus eliminating the effect of a change of diathermancy, the last term in (6) can be neglected. Where first assumed values are such that the corrections required are large, a second approximation may be necessary.

299. From these conditions the constants A and p have been determined from two series of half-hourly observations made with the Arago-Davy actinometer, which, together with the different steps in the work and the results, are given in detail in Professional Paper No. XIII. A similar work has likewise been done by Professor Winslow Upton with several series of observations made on the expedition to Caroline Island to observe the solar eclipse of May 6, 1883,[11] with an Arago-Davy actinometer constructed by Mr. Green. In all these it is shown that the expression of (101), § 50, satisfies the observations with small residuals, at least where forenoon and afternoon observations are combined, so as to eliminate the effect of changes in the diathermancy during the day. The values of the constants are also satisfactory, the value of A obtained being only a little greater than those obtained by Crova with his dynamic actinometer, and those which would be obtained by Herschel's and Pouillet's if the observations were made by rejecting the first minute of exposure after reversals from sunshine to shade, and *vice versa*, as Crova did. Of course the value of p differs for different states of the atmosphere and is only a constant while this remains the same. The value of c in (4) was first determined from special observations by means of equations of conditions formed from (95), § 99, for each pair of the thermometers, and the value of B was that determined by Pouillet from the experiments of Dulong and Petit.

It is readily seen from an inspection of the equations of condition that they are not favorable for an accurate determination of these constants, since the two constants are somewhat complementary to each other, that is, the conditions for all altitudes can be satisfied nearly as well by a considerable increase of the one and a corresponding decrease of the other, or *vice versa.* Hence somewhat discordant values of A are always obtained from different equations of condition, with pretty large probable errors where the constants are determined by the method of least squares from a number of such equations.

The value of the constant B adopted in the formulæ (4) must be regarded, for reasons given in § 76, as somewhat uncertain and only approximate. It has also been assumed that the amount of solar heat reflected and observed by the glass inclosure is equal to that of its own radiations reflected back to the thermometer from the glass inclosure, both being about the one-tenth part. Both these require special and accurate experiments for their accurate determination. More experiments are also required upon the rate of cooling, such as those of Dulong and Petit, to determine whether different values of the constant B may not be required for very slight differences in the tension of air where there is very nearly a perfect vacuum. Different black-bulb thermometers do not give exactly the same indications, in some cases the differences being considerable. This may be due to imperfect lamp-black coatings, or a more nearly perfect vacuum in some cases than others.

If, however, these uncertainties and imperfections cannot be determined and overcome directly, so as to make the indications of the static actinometers as reliable as those of the dynamic, it does not at all impair their value. In the expression of (3) there is only one constant to be determined, assuming that the value of μ, as obtained by Dulong and Petit, must hold through the short range of temperature observations required in actinometric measures. The value of this constant being so determined as to make the indications of any pair of thermometers in the Marie-Davy actinometer agree with the intensities of solar radiation determined by dynamic actinometers, or in any other way, then this very convenient apparatus can be used instead of the others, which are comparatively very inconvenient, just as the psychrometer is used, on account of its convenience, instead of the comparatively very inconvenient dew-point and volume hygrometers. Or the value of B in (4) may be determined for the black-bulb thermometer by observing its rate of cooling *in situ*, that is, within the glass envelope, and then c and m being determined from observation as already explained, the value of K becomes known for any pair of thermometers. By this method any imperfection of the lamp-black coating, the effect of reflected rays from the inclosure back again to the radiating bulb, or conduction of heat by the stem of the thermometer, or imperfection of vacuum, unless it should affect sensibly the value of μ, is taken into account. The solar intensity then given by the actinometer should be that which penetrates through the glass inclosure to the bulbs, to which must be added the intensity lost in passing through the glass. This can be determined, at least approximately, by observing the effect upon the indication of a black-bulb thermometer in calm air of a similar plate of glass interposed between the thermometer and the sun.

Retard of thermometers.

300. In all observations with either the black bulb *in vacuo* or the Marie-Davy actinometer, an account must be taken of the retard, as given by (84), § 91. This for a black-bulb of 1^{cm} in diameter and mean air temperature of 30°, we have seen, § 91, amounts to 6.8 minutes, and for a bright bulb of the same diameter and for the same mean air temperature, it is greater in the ratio of unity to the radiating power of glass, that is, § 77, of 1 to 0.83. In the case of the bright bulb, therefore, it would be about 8 minutes. This, in the gradual diurnal change of normal temperatures, does not have much effect upon the indications of the thermometers, especially at and near the time of the maximum diurnal temperature, the effect being greatest in the forenoon and afternoon when the rate of change of temperature, either increasing or decreasing, is the greatest. But in the rapid changes of temperature of very short periods the indications of all static actinometers are not reliable, since they are based upon the assumption of a sensibly static equilibrium of temperature, which does not take place in the case of rapid changes of temperature, except at and near the times of maxima and minima. In the case of very sudden abnormal changes, therefore, as where the sun is partially or wholly obscured at times by clouds, the indications are not reliable until a considerable time has elapsed after the end of the obscuration.

An example of the effect of the retard in case of great changes of intensity within a short period is afforded by the intensities of solar radiation given by observations made by Professor Upton on Caroline Island during the total eclipse of the sun on May 6, 1883.[11] The following table contains the intensities deduced from observations made at the times given during the eclipse, and also for a certain number of times before and after total phase, determined approximately, so that corresponding phases of the eclipse before and after central totality shall have equal portions of the uneclipsed radiating disk. The corresponding intensities for these times are obtained, as well as possible, from the others by interpolation. Unfortunately small clouds passed over the sun's disk at times, which caused some irregularities in the intensities, and these also gave rise to uncertainties in the interpolations. The clouds noted are at 10.40 and 11.05 to 11.20 inclusive; at 11.45, 12.10 to 12.25 inclusive; 12.50 and 13.05. Some of these, however, do not seem to have affected much the regularity of the results.

Time.	Observed intensity.	Time.	Observed intensity.	Time.	Interpolated intensity.	Time.	Interpolated intensity.
h. m.		*h. m.*		*h. m.*		*h. m.*	
10 5	1.58	11 45	....	10 4	1.56	11 35	0.00
10 10	1.58	50	0.07	11	1.52	43	0.09
15	1.54	55	0.16	18	1.45	51	0.20
20	1.49	12 0	0.21	25	1.34	12 0	0.36
25	1.46	5	0.29	32	1.11	8	0.45
30	1.38	10	0.43	39	0.85	17	0.63
35	1.30	15	0.48	47	0.83	26	0.94
40	1.10	20	0.39	54	0.76	35	1.08
45	0.80	25	0.69	11 2	0.58	44	1.26
50	0.85	30	0.84	10	0.47	53	1.39
55	0.84	35	0.99	18	0.23	13 2	1.46
11 00	0.79	40	1.08	26	0.10	11	1.53
5	0.69	45	1.07	35	0.00	20	1.56
10	0.57	50	1.24	Sums.	10.80		10.95
15	0.48	55	1.31				
20	0.49	13 00	1.38				
25	0.20	5	1.46				
30	0.11	10	1.44				
35	0.10	20	1.49				
40	....	20	....				

The beginning, middle, and end of the eclipse were at the times $10^h\ 4^m$, $11^h\ 35^m$, and $13^h\ 20^m$, respectively. The times given in the last columns before and after the middle of the eclipse represent the corresponding times of equal phase, before and after, as nearly as they could be obtained from the times of beginning, middle, and end simply. By correcting the observed intensities for a retard of 7.5 minutes, it is seen that the sums of the intensities before and after the middle of the eclipse, and corresponding to equal phases of the eclipse, are very nearly equal, being 10.80 and 10.95, respectively, showing that there was sensibly a retard in the indications of the thermometer of that amount of time.

As the intensities depend upon the differences of the indications of the two thermometers, the greater retard of the bright thermometer increases the effect and so diminishes the retard in the intensities in the ratio of 1 : 0.83. Hence in the case of bulbs 1^{cm} in diameter the retard in the intensities should be $6.8^m \times 0.83 = 5.6^m$.

The estimated diameter of the bulbs in the actinometers used in the observations during the eclipse is 1.2^{cm} and the air temperature very nearly 30°. Hence the amount of the retard in the intensities in this case should be 6.7^m, being, where the air temperatures are the same, by (84) § 91, in proportion to the diameters.

The retard of the thermometers is important, not only in static actinometers, but likewise in air thermometry, unless the thermometers are ventilated. Where thermometers are exposed in the open air, of course the effect is less, but even then it is often considerable where there are sudden and rapid changes and the air is very calm. The thermometer does not follow closely after these changes.

It has been shown that the effect of ventilation is the same as that of an increase of radiating power, and that its effect is shown approximately in all cases by putting $(r+kv)$ instead of r simply, v being the velocity of ventilation and k an unknown constant. With this change in (84), § 91, it is seen that the effect of ventilation is to decrease the amount of retard R, and that when v is infinite R vanishes. The retard, therefore, can be made to sensibly vanish by ventilation where the thermometers are exposed in the air and not *in vacuo*.

Hygrometry.

301. The amount of aqueous vapor in the air varies very much in different places at the same time, and in the same place at different times. The range of variation may be from almost perfect dryness up to a state of saturation, which, according to Table X, may amount in very warm climates to one-twentieth or more by volume of the atmosphere. The general problem of hygrometry, according to Regnault, "consists in determining the quantity of aqueous vapor existing at any time in a given volume of air, and the relation between this quantity and that which air would have if it were perfectly saturated."

There are several methods of determining the tension of the aqueous vapor of the air and the fraction of saturation. It may be done chemically by weighing the quantity of humidity which is absorbed from a given volume of air passed through tubes which leave it perfectly dry. They are also obtained with some less degree of accuracy by the well-known dew-point hygrometers of Daniell[12] and Regnault.[13] More recently, also, Schwackhöffer's *volume-hygrometer*[14] has been used, which is said to give very satisfactory results. By each of these methods the amount of aqueous vapor in the air at any time can be pretty accurately determined, and one of them should be used in all kinds of physical research where great accuracy is required, but they are all too inconvenient and expensive to be used in the daily observations of the hygrometric state of the atmosphere.

A far less accurate means of determining the hygrometric state of the air is founded upon the indications of hygrometers formed of organic substances which are lengthened by humidity. Of these, the most noted is Saussure's hygrometer,[15] the indications of which depend upon the expansibility of hairs with increase of humidity when they are well cleaned and prepared by a certain chemical process. Regnault[16] made a great many comparisons of these instruments, in some cases of instruments made with hairs prepared in the same way, and in others of instruments made by different persons with hairs prepared in different ways, to see if they were comparable. The conclusions at which he arrived are, "that hygrometers constructed with hairs of the same kind, divested of grease in the same operation, do not exactly keep pace, but that, nevertheless, they do not differ so much in most of the observations as to be regarded as not comparable." He also found that hygrometers constructed with hairs of different natures and prepared in different manners may present very great differences in their indications, even when their fixed points agree. This hygrometer, therefore, is not now used in meteorological observations or for any purpose where accuracy is required, at least unless it is very frequently compared with some standard, but it is convenient and useful in hospitals and hygienic institutions to indicate approximately the humidity of the air.

302. The *wet and dry bulb hygrmometer*, usually called *the psychrometer*, and first suggested by Gay-Lussac, although not of very great accuracy, is now mostly used on account of its convenience in the ordinary daily observations of the hygrometric state of the air. Subsequently M. August,[17] a German physicist, investigated this question and published several interesting memoirs, in which he sought to establish upon theoretical principles the formula according to which may be calculated the tension of aqueous vapor existing in the air from the temperatures indicated by the dry and moist bulb thermometers in the air. The fundamental hypothesis adopted by M. August in his theory was *that all the air which supplies heat to the moist thermometer falls to the temperature indicated by the latter, and is completely saturated with humidity.*

Let

t, t' = the temperatures, respectively, of the dry and wet bulb thermometers;

f' = the tension of aqueous vapor in saturated air of temperature t_1;

x = the tension of vapor in millimeters existing in the air;

H = the height of the barometer.

The formula obtained by M. August is

$$x=f'-\frac{0.558(t-t')}{640-t'}H \tag{1}$$

This formula was subsequently changed by Regnault, by adopting more accurate values of the physical constants used, to

$$x=f'-\frac{0.429(t-t')}{610-t'}H \tag{2}$$

Regnault, however, thought that August's fundamental hypothesis was not strictly correct, and thought it more prudent to employ theoretical considerations only in the investigations of the *form* of the function and to determine afterwards the constants by experiments made in fixed conditions.

From numerous comparisons with the formula of results obtained by himself by weighing the amount of vapor absorbed from the air, and also of results obtained by M. Marié and M. Izarn with a condensing hygrometer, the former at St. Etienne under a barometric pressure of 705^{mm}, and on Mount Pila under a pressure of 655^{mm}, and the latter in the Pyrenees, he found that if the numerical coefficient 0.429 were changed to 0.480, the formula would represent all the observations best. Hence the formula, used mostly up to the present time,

$$x=f'-\frac{0.480\,(t-t')}{610-t'}H \tag{3}$$

This coefficient gave almost a perfect coincidence between the calculated results and those found by direct weighing where the fractions of saturation exceeded 0.40, but for weaker fractions of saturation the vapor tension given by the formula was found to be too large.

For freezing temperatures the denominator in the last term became by the theory $689-t'$, since 79 heat units are required to melt a unit by weight of ice, and consequently $689-t'$ to convert ice to vapor of temperature t'.

Putting the preceding formula into the form,

$$x=f'-A(t-t')H \tag{4}$$

Regnault gives in his second memoire[18] the results of many experiments made under different circumstances, and the determination of the value of the constant A in this formula which gave the best agreement between the formula and the experiments under the different circumstances. These were as follows:

In a closed room of 100 cubic meters	0.00128
In the large amphitheater of the College of France with the windows closed.	0.00100
In the same with two opposite windows open	0.00077
In a large square paved court of the college	0.00074
In a long court planted with trees	0.00100
In the same, with psychrometer in sunshine	0.00090
With the wet-bulb thermometer below zero	0.00075

M. Angot[19] has determined the values of A in the same formula from 3,388 comparisons of the psychrometer and condensing hygrometer for temperatures of the wet-bulb thermometer above 0°, and from 282 when the same thermometer was covered with a coat of ice. These comparisons were mostly of observations made at the lower station of the observatory of the Puy-de-Dôme, altitude 390^{m}, but 205 were made at the summit station, altitude $1{,}470^{m}$, for temperatures of t' above 0°, and 27 for temperatures below 0°.

In classifying the results for different ranges of the value of $(t-t')$ in the formula, he found that the value of A in the formula required to satisfy the observations decreased as $(t-t')$ increased. These values of A ranged from observations made at Paris from 0.000947 for $(t-t')=3.4°$ to 0.000786 for $(t-t')=11.9°$, and at the lower station of the Puy-de-Dôme from 0.001022 for $(t-t')=0.53°$ to 0.000705 for $(t-t')=6.77°$.

From a comparison of the results given by seven psychrometers with those of Daniell's dew-point hygrometer, in a series of 36 experiments, Käemtz[20] obtained for the average of the constant, when the formula is put in the form,

$$(5) \qquad x=f-'a(t-t')$$

$a=0.6527$. This gives for the preceding form (4), $A=0.000806$.

303. It is seen from the differences of the preceding values of A with closed and open windows, that the value of this constant required for air at rest is much greater than in the case of air in motion. Even before the time of Regnault's researches Belli[21] had shown that the indications of the psychrometer depend very much upon the velocity of the wind, especially where this velocity is small, but less where the velocity is greater, and that they become nearly independent of the variations of the velocity where it is very great.

He subsequently devised a psychrometer in which the dry and wet cylindrical reservoirs were placed in a tube through which a current of air was drawn by means of a bellows. The air first passed over the dry and then the wet reservoir. A very ingenious part of the arrangement was to have the wet cylindrical reservoir surrounded by a larger cylindrical tube which was covered with a wet cloth in the same manner as the thermometer, so that it cooled down by evaporation to the temperature of the wet thermometer. Hence there was almost a complete inclosure around the wet cylindrical reservoir of the same temperature, and consequently all effect of radiation from a warmer inclosure was cut off.

The values of A required for sling and ventilated psychrometers are found to be in all cases smaller than 0.0008, which has been generally used, and which may be supposed to be the value required for the average of all degrees of ventilation in observations at different times and under different circumstances. M. L. Doyère[22] has obtained from 21 experiments in the shade the mean value of $A=0.000687$. These experiments were made at temperatures ranging from 11° to 28°, and hence none of them at low temperatures. The values of $(t-t')$ ranged from 1° to 10°. The experiments in the sunshine gave in all cases values greater than this. Hence he concludes that the psychrometer should be slung in the shade, where not only the direct, but likewise mostly the reflected, solar rays are cut off.

The most important experiments in determining the relations between the values of the constant A and the velocity of ventilation have been made by Sworykin.[23]

The comparisons of the ventilated psychrometers were made with a volume-hygrometer of Schackhöffer and one of Alvord's condensation-hygrometers. The following table contains the values of A, determined from these comparisons, for two hygrometers with different kinds of reservoirs, corresponding to each of the wind velocities contained in the first column:

Velocity of wind—meters per second.	Values of A for a psychrometer with—			
	Spherical bulb 10^{mm} in diameter.		Cylindrical bulb 8^{mm} in length and 4^{mm} in diameter.	
	Observed.	Computed.	Observed.	Computed.
0.0	....	0.001658	....	∞
0.2	0.001120	.001084	0.000854	0.000893
0.4	.000920	.000923	.000767	.000785
0.6	.000845	.000856	.000730	.000741
0.8	.000800	.000810	.000713	.000715
1.0	.000774	.000780	.000700	.000699
2.0	.000712	.000711	.000670	.000677
3.0	.000687	.000686	.000657	.000659
4.0	.000673	.000678	.000646	.000646
5.0	.000664	.000664	.000642	.000644
6.0	.000656	.000658	.000640	.000643
10.0	....	.000647	....	.000637
∞	....	.000630	....	.000630

These numbers are read off from the graphical representation of his results. Those for the psychrometer of smaller reservoirs are read from the broken-line curve, which, on account of its greater analogy to that of the other psychrometer with large spherical reservoir, is supposed to represent the most probable values of A.

For the computed values of the table the formulæ of (13) and (17), § 304, in the case of the spherical bulbs, and (13) and (19) in the case of the small cylindrical ones.

From this table it is seen that the value of A is comparatively very large for small wind velocities, and the changes of value for given changes of velocity very great. Consequently there is great uncertainty in any results obtained with an unventilated psychrometer with the air very nearly at rest, since a change from 0.2 to 0.4^{m} per second changes the value of A required in the formula about one-fifth part. With a large ventilation, however, a little variation in the velocity of the wind has comparatively very little effect, and there is evidently a limit toward which the value of A tends beyond which it cannot go, even with infinite ventilation.

It is also seen that different bulbs require different values of A, and this is especially the case when there is little ventilation. As ventilation is increased the values of A required for the two kinds of bulbs seem to approximate to the same limit, below which they cannot go, and consequently to approximate to each other. In a well-ventilated psychrometer, therefore, the uncertainties arising from differing velocities of ventilation, as well as those from differing sizes and shapes of the bulbs, both, in a great measure, disappear. These experiments, of which the results are given in the preceding table, were made at temperatures from 15° to 20°, and with ranges of $(t-t')$ from 5° to 8°.

304. The psychrometer formula, although originally deduced theoretically by M. August from his adopted hypothesis, § 302, has been regarded mostly as an empirical formula, if not in form, at least in the value of the constant A. But since the theoretical researches of Maxwell,[24] and the theoretical and experimental researches of Stefan,[25] in the kinetic theory of gases and their interdiffusion, the formula can be placed more upon a theoretical basis. Based upon the theory of the conduction of heat and interdiffusion of gases, as deduced from the kinetic theory of gases, the fundamental principle now, instead of that adopted by August, is, that as much heat must be received by the wet bulb by conduction from the air and by radiation from its inclosure as is necessary to supply the latent heat of evaporation to the aqueous vapor as it flows away by diffusion. The rate of conduction of heat to the wet bulb depends upon the gradient of decreasing temperature between the general temperature of the air and that of the wet bulb, and the rate of diffusion of the vapor depends upon the gradient of decreasing vapor tension between the surface of the wet bulb and that of the surrounding air. It is evident that both these gradients depend upon the temperature of the wet bulb, for the higher this is the smaller is the temperature gradient and the greater the vapor-tension gradient. The temperature of the wet bulb, therefore, must be such that the temperature gradient, with the known coefficient of conduction, will supply as much heat to the wet bulb as is required to supply the latent heat of evaporation to as much vapor as can flow away, with the known diffusion constant, by virtue of the vapor-tension gradient, which also depends upon this temperature.

In the theory of the psychrometer let us put

H=the rate with which heat is conducted to the wet-bulb thermometer;

Q=the rate by volume with which aqueous vapor is diffused through the air;

h=the rate with which heat is received and absorbed by the wet-bulb thermometer from the radiation of its inclosure above what it radiates;

r=radial distance from the center of wet bulb;

P=the pressure of the air;

p=the tension of the aqueous vapor;

θ=the air temperature;

r_0=the value of r at the distance at which the temperature and vapor tension are not sensibly disturbed by evaporation;

r_1=the values of r at the surface of the bulb;

p_0, p_1=the values of p at the distances r_0 and r_1 respectively;

θ_0, θ_1=the temperatures at these distances respectively;

a=the sectional area through which conduction and diffusion takes place;

S=the specific heat of air;

ρ=the normal density of dry air;

σ=the relative density of aqueous vapor compared with dry air;

L=the heat of evaporation.

For the rate with which heat passes through the sectional area a, we have

$$H=aK\frac{d\theta}{dr}$$

in which K is the constant of conduction, that is, the rate with which the heat passes when the sectional area a is unity, and the temperature gradient $D_r\theta$ is unity.

As in the case of the flow of liquids and gases, so in this case, the condition of continuity must also be satisfied. In the case of a spherical bulb the value of a is the surface of a sphere of radius r, and hence we have

$$a=4\pi r^2 \tag{6}$$

Substituting this value of a in the preceding expression of H, and integrating from $r=r_0$ to $r=r_1$, we get

$$H=4\pi K\frac{r_1 r_0}{r_0-r_1}(\theta_0-\theta_1). \tag{7}$$

According to the theoretical researches of Stefan in the diffusion of gases and in evaporation, we have

$$Q=-aD\frac{1}{P-p_1}\cdot\frac{dp_1}{dr}=aD\frac{d\log(P-p_1)}{dr}$$

in which D is the diffusion coefficient of aqueous vapor through air.

Substituting for a its value in (6) and integrating as before, we get

$$Q=-4\pi D\frac{r_1 r_0}{r_0-r_1}\log\frac{P-p_0}{P-p_1}=4\pi D\frac{r_1 r_0}{r_0-r_1}\,\frac{p_1-p_0}{P} \tag{8}$$

the latter form being obtained from the preceding by developing the logarithm by a well-known formula, and neglecting quantities of the order of p_0 and p_1 in comparison with P_0.

The rate with which heat is received by the wet bulb by radiation from its inclosure in excess of the heat radiated from it, for each unit of surface of the wet bulb, is, § 76, $Be\,(\mu^{\theta_1}-\mu^{\theta_0})$ in which the relative radiating power of the wet bulb is here denoted by e. We shall therefore have for the whole surface of the bulb, which by (6) is $4\pi r_1^2$,

$$h=aBe\,(\mu^{\theta_0}-\mu^{\theta_1})=4\pi r_1^2 Be\times .0077\theta_1\,(\theta_0-\theta_1) \tag{9}$$

In this expression it is assumed that the inclosure is perfect and of the temperature of the surrounding air, which is not strictly the case where the wet bulb is exposed in the open air; for a perfect inclosure is one from which the body receives with a static temperature as much heat by radiation and reflection from its surroundings as it radiates, which, we have seen, § 86, is not the case where some of the radiated heat passes out into space. In such a case the value of θ_0 required in (9) is a little less than the air temperature.

In cloudy weather, however, or where there is a screen between the wet bulb and the clear sky, the value of θ_0 required is that of the air temperature and other surroundings, or rather the average where they are of different temperature.

The rate of loss of heat by the wet bulb, being the latent heat of evaporation in the aqueous vapor which is diffused away from it, is $L\rho\sigma Q$. Hence by the fundamental principle of the diffusion theory of the psychrometer, given above, we have

$$L\rho\sigma Q=H+h \tag{10}$$

This equation, by means of (7), (8), and (9), becomes,

$$p_1-p_0=\frac{PK(\theta_0-\theta_1)}{L\rho\sigma D}+\frac{.0077\mu^{\theta_1}Ber_1(r_0-r_1)(\theta_0-\theta_1)}{L\rho\sigma Dr_0}$$

in which the last term expresses the effect of radiation. This may be put into the following form:

$$p_0=p_1-A(\theta_0-\theta_1)P \tag{11}$$

in which

$$\begin{aligned} A&=\frac{PS}{L\sigma}\left(\frac{K'}{D}+c\mu^{\theta_1}\right)\\ K'&=\frac{K}{\rho S}\\ c&=\frac{.0077Be}{\rho SD}\cdot\frac{r_1(r_0-r_1)}{r_0} \end{aligned} \tag{12}$$

According to Regnault $\sigma=0.622$ and $L=606.5-0.695\theta_1$, neglecting the terms of the higher powers of θ_1, since their value in psychrometry is always small. According to Stefan, $D=0.18$ and $K=0.000055$; and hence, with the known values of $\rho=0.00129276$ and $S=0.2375$, we have $K'=0.18$. We can also put $\mu^{\theta_1}=1+.0077\theta_1$.

The values of K' and D in (12_1) are not independent of the temperature, but both, by the kinetic theory of gases, increase with increase of temperature, but neither the theory nor observation gives certainly the law of increase, but they seem to indicate that the rate of increase with increase of temperature is greater for D than for K and consequently K'. The value of A, therefore, on this account would decrease a little with increase of temperature. The value of L is also a function of the temperature which makes the value of A increase with increase of temperature. Both these effects are very small and tend to counteract each other, and we shall therefore assume that A is independent of temperature. And this was indicated by Sworykin's experiments.

With the preceding values of the constants, neglecting the temperature term in that of L, we get from (12)

$$(13) \qquad A = 0.000630\,[1+c(1+.0077\theta_1)] \qquad\qquad c = 2.66e\,\frac{r_1(r_0-r_1)}{r_0}$$

The part of the expression of A depending upon c, arising from radiation, is in proportion to the relative radiating power of the thermometer bulb, which, where covered with wet muslin, is perhaps but little below unity.

Where the temperature of the wet bulb is below zero, the heat of liquefaction, 79.25°, must be added to that of the evaporation of water in order to have that of the evaporation of ice, and then we have $L = 685.75 - 0.695\theta$. With this value of L instead of the preceding, omitting the part depending upon θ_1 for reasons already stated, we get

$$(14) \qquad A = 0.000557[1+c(1+.0077\theta_1)]$$

The value of A, however, required to satisfy the experiments with temperatures of the wet bulb below zero in Sworykin's experiments were apparently the same as in the case of the temperature above zero, but the values of $(t-t')$ in the former are always so small, and consequently the probable errors of A so large in any experimental determination of it, that it would require a good many experiments to show decisively that the value of A in the two cases should be the same. According to the diffusion theory of the psychrometer they cannot be so.

In a perfectly quiet atmosphere the temperature and vapor-tension gradients would extend to infinity, though they would sensibly vanish at a finite and not very great distance from the wet bulb. In this case, therefore, the expression of c becomes

$$(15) \qquad c = 2.66er_1$$

305. If we had a wet bulb with cells of small depth over its surface similar to those of a honey-comb, as represented in section by the adjacent figure, and the bulb was subject to ventilation, then the gradients would extend only to the mouth of the cells, and the preceding conditions, both of those depending upon the gradients and that of continuity, would hold from the surface of the bulb or bottom of the cell to its mouth, and in this case, therefore (r_0-r_1), would become the depth of the cells. At the mouth of the cells with ventilation the air would have the temperature and humidity of the air generally. The cells would be similar to those used by Stefan in his experiments in evaporation, and the laws of action would be the same, the heat required for the evaporation being conducted into the bottom of the cell by the temperature gradient, and the vapor, as fast as formed, diffused out by means of the vapor-tension gradient.

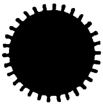

FIG.

In the case of ventilation without such an arrangement, the condition of continuity (6) would not hold, since heat would be supplied and vapor carried away by ventilation at all distances from the surface of the bulb, and the rate of flow would depend only in part upon the gradients of temperature and vapor tension. We may, however, suppose that the effect is the same as would take place in case of cells of some unknown depth, and that in all the varying velocities of ventilation this unknown depth is inversly as the velocity of ventilation. If we therefore put in (13_2)

$$(16) \qquad k=2.66e(r_0-r_1)v$$

k will be a constant, but unknown because the value of (r_0-r_1) is unknown. With this (13_2) becomes

$$(17) \qquad c=\frac{r_1}{r_0}\cdot\frac{k}{v}=\frac{2.66er_1k_u}{2.66er_1v+k}$$

If the theory and the preceding hypothesis are correct a value of k and corresponding value of c can be found, which in (13_1) will give the observed values of A for each of the different velocities of ventilation in the table of § 303. It is seen from the expression of A that there is a small term in the effect due to radiation depending upon the temperature θ_1. Assuming that this for the average temperature of the experiments is 15°, we get $(1+.0077\theta_1)=1.115$. With this value in (13_1) if we assume in (17) $k=0.25$, and with the value $r_1=0.5$, we get the computed values in the table of § 303 for the psychrometer with the large spherical bulb. The first of the computed values is given by (13_1) with the value of c obtained from (15), or from (17) by putting $v=0$.

306. In the case of a cylindrical bulb of infinite length, and approximately where it is so long that the area of the surface of the ends can be neglected in comparison with the whole surface, instead of (6) we have

$$(18) \qquad a=2\pi rl$$

in which l is the length of the bulb, r being the radius as before. This for the surface of the cylindrical bulb becomes $a=2\pi r_1l$. The first of these values of a being substituted in the differential expressions of A and Q, which give (7) and (8) by integration, and the latter value in (9), and proceeding precisely as in the previous case, we get the expression of (13_1), but instead of (13_2) we get in common logarithms

$$(19) \qquad c=\frac{266er_1}{\mathrm{M}}\log\frac{r_0}{r_1}=0.612r_1\log\frac{r_1}{r_0}$$

Adopting here also the hypothesis of (16), and assuming in this case $k=0.11$, we get from (16) the values of (r_0-r_1) and also the values of $(r_0:r_1)$, for the smaller psychrometer in the preceding table with the cylindrical bulb, in which $r_1=0.2$. With the values of c in (19) obtained with these values of $(r_0:r_1)$ and the previous assumed value of

$1+.0077\theta_1=1.115$, (13_1) gives the computed values of A in the table of § 303. These computed values, as in the previous case, agree very well with the experimental ones of the table, for these latter, from the fewness of the experiments and their nature, have necessarily considerable probable errors, especially for the small velocities of ventilation.

307. From the preceding results it is seen that with the values of c in (13_2) in the case of spherical bulbs, and in (19) in the case of cylindrical ones, obtained upon the hypothesis of (16) with a proper value of the constant k, (13_1) gives very nearly the values of A for the different velocities of ventilation. Hence with a value of k determined for any one given velocity, the value of A becomes known for all others. We have seen, however, that very different values of the constant k are required for the large spherical bulb and the small cylindrical one, the former requiring $k=0.25$, and the latter, $k=0.11$. It is also probable that bulbs of different sizes of the same kind would require different values of k, so that each kind, and also the different sizes of the same kind, require k to be determined by experiment for some one value of v.

Since the value of c in (13) depends upon radiation from the warmer inclosure of the wet bulb, and its value varies with different amounts of ventilation, with an arrangement like Belli's, § 303, all these effects would be cut off and the same value of A would be required for all degrees of ventilation, and this value would be that required in the case of infinite ventilation. The only advantage of much ventilation then would be to quickly reduce the temperature of the wet bulb to its static state.

The expression of A (12), if we put $K'=D$ according to Stefan's experiments, and neglect the term depending upon radiation, is the same as that deduced by August from his assumed hypothesis, § 302_1 neglecting very small effects in his expressions. It is, therefore, on the merely accidental agreement of the values of K' and D that August's theory, with the improved constant of Regnault, gave results according very nearly with observation and with the diffusion theory of the psychrometer. According to this theory, also, August's fundamental hypothesis, is not correct; for since by this theory heat is conducted to the wet bulb by means of a temperature gradient, and the aqueous vapor is diffused away by means of a vapor-tension gradient, the temperature increases and the vapor tension decreases with increase of distance from the wet bulb, and as only the infinitely thin stratum of air in contact with the bulb is supposed to be saturated, the air at some distance from it cannot be saturated, and so all the air which supplies heat to the moist thermometer is not completely saturated with humidity, as supposed in August's hypothesis.

With other ratios between K' and D than that of unity, the numerical coefficient in (13_1) would be different. For instance, if it were 0.77, supposed by Maxwell to be the probable value, this coefficient would be 0.000482 instead of 0.000630. But it is readily seen from Sworykin's graphical representation of the values of A for different velocities of ventilation,[23] as well as from the numerical values given in the table of § 303, that no amount of ventilation would bring the value of A as affected by radiation down to this much smaller value. The experimental and computed values of A in the table, in the case of both psychrometers, approximate to the value of 0.000630 obtained from Stefan's values of K' and D, and so these experimental values are corroborated by Sworykin's experiments of a very different kind.

Reductions of barometrical observations.

308. It is seen from (17) and (18), § 13, that there are three corrections to be applied to a barometric reading in order to get the true barometric pressure; the first depending upon the latitude λ, the second upon the altitude h, and the third upon the coefficient of the relative expansions of mercury and of brass, β, where the barometer has a brass scale extending down to the base of the mercurial column.

With the value of r' in (40), § 16, we get very nearly

$$(1)\qquad \frac{2(h-h')}{r'}=0.0000003(h-h')$$

in which h' is the value of h at the lower station, where it is not at sea-level.

If we put

β'=the coefficient of the vertical expansion of mercury as read from a brass scale;

l=the coefficient of the linear expansion of brass;

we shall then have

$$(2)\qquad \beta'=(\beta-l)$$

According to Regnault, the coefficient of the expansion of mercury β at the temperature of 15°, which is about a mean temperature in meteorological observations, is 0.00017978. The error arising from adopting the mean instead of the varying coefficient, even in extreme cases within the usual range of observations, is generally not more than 0.02^{mm}, and hence of no consequence. The coefficient of the linear expansion of brass, according to Lavoisier and Laplace, is 0.00001878. Hence we get

$$(3)\qquad \beta'=0.000161$$

which must be used instead of β where the readings are from a brass scale. If the readings were from a true scale without expansion, we should have $l=0$, and $\beta'=\beta$.

With this value instead of β in (18), § 13, we get for the reduced barometer reading, or true barometric pressure,

$$(4)\qquad P=\frac{B}{(1+.002606\cos 2\lambda)[1.0000003(h-h')](1+.000161t)}$$

P' being the value of P at altitude h'

In this it is supposed that the standard temperature of the scale is 0°. In English barometers it is 62° F.=16°.67 C. In this case it is necessary to subtract from the reading from a brass scale the expansion of the brass corresponding to 16.67°, which is

$$(5) \qquad 16.67lB = 16.67 \times 0.00001878B = .000313B$$

In English measures, with scale at standard temperature of 62° F., and h expressed in feet, by changing the numerical coefficients inversely as the units of measure and obtaining the term of (5), we get

$$(6) \qquad P' = \frac{B - .000313B}{\left.\begin{array}{c}[1+.002606 \cos 2\lambda]\,[1+.000000091(h-h')] \times \\ [1+.0000895(t-32^\circ)]\end{array}\right\}}$$

$$= \frac{B}{\left.\begin{array}{c}[1+.002606 \cos 2\lambda]\,[1+.000000091(h-h')] \times \\ [1+.0000895\,(t-32^\circ)]\,[1+.000313]\end{array}\right\}}$$

$$= \frac{B}{\left.\begin{array}{c}[1+.002606 \cos 2\lambda]\,[1+.000000091\,(h-h')] \times \\ [1+.0000895(t-28.5^\circ)]\end{array}\right\}}$$

By means of the tables in the appendix we get from this

$$(7) \qquad \log P' = \log B - (\text{Table IV} + \text{Table VI (Arg. } h-h') + \text{Table VII})$$

in which B is the barometric reading, uncorrected even for temperature.

For sea-level and parallel of 45°, we have of course only the last table of (7′), belonging to the reduction to the temperature of 32° F., but in most cases all of these corrections have to be applied to the barometric reading to obtain the true barometric pressure at the place of observation.

In what precedes, it is assumed that we have a perfect barometer, with the scale so adjusted as to correct for capillarity, and needing no corrections for errors of graduation.

309. Where barometric observations are made at different altitudes, they are not comparable with one another in the determination of horizontal barometric gradients until they are all reduced to the same level, which is usually that of sea-level, and the reduction is then called *reduction to sea-level.* If the barometric readings have been already corrected as above for the variations of gravity with variations of latitude and altitude from the standard gravity of the parallel of 45° and of sea-level, and also for temperature, the practical formula for reduction to a lower plane of altitude h' is that of (49), § 18. The value of the first member having been computed, and the value of P being known, that of P' corresponding to the altitude h is readily obtained. In the case of reduction to sea-level, we have $h'=0$.

The effect of the correction of P for altitude, however, can be conveniently taken into the formula, and then it is only necessary to correct for latitude and temperature before using the formula Putting

$B_1 = B$ corrected for latitude and temperature only but not for altitude,

we shall have

$$(8) \qquad \log B_1 = \log B - (\text{Table IV} + \text{Table VII})$$

and from (6)

$$\log P' = \log \left[B_1 - \frac{2(h-h')}{r'} \right] = \log B_1 - \frac{2M(h-h')}{r'}$$

in which, by (4) and (1) the last term is the correction of log P for altitude. With this value of log P in (37), § 15, and the value of $q = 1 + 0.002606 \cos 2\lambda$ we get

$$(9) \quad \log \frac{P'}{B_1} = \frac{M.}{l} \frac{(h-h')\left(1-\frac{2l}{r'}\right)}{\left[1+\frac{1}{2}\alpha(\tau'+\tau)+0.189\,(e'+e)+\frac{h'+h}{r'}\right](1+.002606 \cos 2\lambda)}$$

$$= \frac{M}{l\left(1+\frac{2l}{r'}\right)} \frac{h-h'}{[1+\frac{1}{2}\alpha(\tau'+\tau)]\,[1+.189(e'+e)]\left(1+\frac{h'+h}{r'}\right)(1+.002606 \cos 2\lambda)}$$

Another transformation for the sake of convenience can be introduced into this formula, where the psychrometer is used to obtain e, by expressing e in a function of B and τ_1, τ_1 being, as in § 304, the temperature of the wet bulb of the psychrometer. From (26), § 14, and (11), § 304, we get

$$(10) \qquad e = \frac{p}{P} = \frac{p_1}{P} - A\,(\tau - \tau_1)$$

Hence we have

$$(11) \quad 1+0.189\,(e'+e) = \left(1+0.189\frac{p'_1}{P_1}\right)\left(1+0.189\frac{p_1}{P}\right) \times$$
$$[1-.189A(\tau'-\tau'_1]\ [1-.189A(\tau-\tau_1)]$$

Since by Table X p_1 is a function of τ_1, we have

$$(12) \qquad 1+.189\frac{p_1}{P} = \varphi(P, \tau_1)$$

a function of P and τ_1. By means of these expressions and the values of l and r' in (40), § 16, we get from (9), putting $H = h - h'$,

$$(13) \qquad \log P' - \log B_1 = \frac{H}{K}$$

in which

$$K = 18445.7 \left[1+.001835\,(\tau'+\tau)\right] \left[\varphi\,(P', \tau')\right] \left[\varphi(P, \tau)\right] \times$$
$$\left[1-189\,A\,(\tau'-\tau_1)\right] \left[1-189A(\tau-\tau_1)\right] \times$$
$$\left[1+.002606 \cos 2\lambda\right] \left[1+\frac{H}{r'}\right] \left[1+\frac{2^{h}}{r^{1}}\right]$$

Putting $A=.0008$, which is the usual value where there is no special ventilation of the psychrometer, we have in feet and degrees Fahrenheit,

$$\begin{aligned}\log K=4.78189&+\log\,[1+.00102\,(\tau'+\tau-64^\circ)]+\log\,\varphi(P',\tau')\\&+\log\,\varphi(P,\tau)+\log\,[1-.000084\,(\tau'-\tau'_1)]\\&+\log\,[1-.000084\,(\tau-\tau_1)\,]+\log\,[1-.002606\cos 2\lambda]\\&+\log\,[1+.000000091\,H]+\log\,[1+.000000182\,h']\end{aligned}$$

These logarithms may be obtained from the tables in the Appendix for any given values of the arguments. Table I gives $+4.78189 \log (1.00102)\,(\tau'+\tau-64^\circ)$. Table II is computed from (12) with the value of p_1, obtained from Table X. Tables II and III have to be entered twice, once with the arguments of the lower level, usually sea-level, and distinguished by an accent, and then with those of the upper level. The arguments of Tables V and VI are given for both English and French units. Table I can be used with French measures by deducting the logarithm 0.51599 and converting the temperature to degrees Fahrenheit. The other tables with temperature arguments can be used by a similar conversion of the arguments.

With the preceding expressions of $\log B_1$ (8) and of $\log K$, (13) gives

$$(14)\qquad \log P'=\log B+R+\text{Table IV}+\text{Table VII}$$

in which

$$(15)\qquad \log R=\log H-(\text{Table I}+\text{Table II (bis)}+\text{Table III (bis)}+\text{Table IV}+\text{Table V}+\text{Table VI})$$

If B has already been corrected for temperature, then Table VII must be omitted in (14). If also the reduction is to sea-level, as is usually the case, then h' vanishes and Table VI in (15) is not needed.

310. It is seen from (11) that the reduction to sea-level requires a knowledge of both the temperature and the hygrometric state of the atmosphere at the earth's surface, and in the formula it is assumed that these decrease with increase of altitude at a regular rate. Usually the reductions required are those of observations made at stations on land of greater or less elevation, or on mountain tops and on plateaus. In such cases it is somewhat difficult to conceive what reduction to sea-level means, since there is no air pressure there. But since what is wanted by such reduction is something comparable with surrounding places where there is no elevation, it cannot mean anything more than the determination of what the barometric pressure would be if there were no mountain, or land surface, above sea-level. Upon this hypothesis, therefore, we do not know the proper temperature and hygrometric state to be used, even for the upper station, for these are different on the plateau or mountain top where the observations are made from what they would be if the plateau or mountain were away, and from what they are in the open air around about at the same altitude as the station. We have here all the difficulties which were found, § 289, in the reductions of temperature to sea-level.

In the case of annual averages the temperature of the air on mountain tops and elevated plateaus is very nearly the same as that in the open air generally at the same altitude, but even in this case there is an uncertainty with regard to the rate with which it is reduced to sea-level, since this, by the table of § 140, determined by comparing the temperature of mountain tops with distant lower stations, seems to differ in different places. It is usual to adopt, however, in this case the rate of about 0.5° for each interval of 100 meters.

In the case of monthly averages, since the mean annual range of temperature is greater on any land surface, even on mountain tops, than in the open air, the observed temperatures are greater in summer and less in winter, than the temperatures required in the reductions to sea-level. The rates, also, as may be seen in the table just referred to, by which the reductions of the temperatures to sea-level should be made vary with the seasons, and are greater in summer than in winter, so that the uncertainties in the values of τ' to be used in the formula of the reduction of the air pressure to sea-level are increased both on account of the uncertainty in the open-air temperature at the elevation of the upper station, and also on account of the uncertainty of the rate by which the reduction of temperature to sea-level should be made. During the spring and fall when the monthly average of temperature does not differ much from the annual mean, and the rate in the reduction to sea-level is about the same, the uncertainty in the reduction of pressure to sea-level is very little more than in the case of annual averages.

311. Mean *diurnal* temperatures are affected by all the abnormal disturbances which cause them to deviate from the monthly averages, and these deviations are much greater on any land surface or mountain top than in the open air, so that on unusually warm days the observed temperatures are higher, and on very cold days lower, than the temperatures required in the reduction of barometric pressure to sea-level, and there is very great uncertainty in the reduction of these abnormal temperatures to sea-level in order to get the lower temperature τ' from the observed temperature τ at the station of observation.

All that has been stated with regard to annual variations is true of the diurnal variations both normal and abnormal. These are greater on any land surface and mountain peak than they are in the open air, and the rates of reduction to sea-level may be very different from that of the mean of the day or of the monthly average. Generally in the morning, especially in clear weather, the air temperature obtained from observation is too low and in the afternoon too high, to be used as the true open-air temperature required in the reductions of barometric pressure to sea-level, and if so used the reduced pressure of the morning is too great, and that of the afternoon too small, to make it comparable with observed pressures of lower stations.

In the reductions, therefore, of barometric pressure to sea-level it is necessary to diminish in some measure the extreme deviations of observed temperatures from the mean, both annual, diurnal, and abnormal, and to use one which is more nearly that of the average; but for this there are no certain rules. These may differ both at different places and at different seasons. Where the station is inland and far from the ocean the deviations of observed temperatures from the true air temperatures are very great, while at maritime stations, especially if the ocean is windward, they are much less. If the observations of pressure at higher stations are to be compared with those of lower ones, some kind of reduction is absolutely necessary, but we have no certain data and rules upon which to base this reduction, and they must necessarily be left somewhat arbitrary and dependent upon good judgment.

312. In view of the great uncertainties in the reductions of barometric pressure to sea-level, it is not worth while to introduce any nice refinements into it, especially in cases of the reduction of individual observations. The best that can be done in these cases is to use the normal average humidity belonging to the given temperature. This is taken in approximately by Laplace's modified temperature coefficient in the formula of (13), which is .002 instead of .001835. For low temperatures this correction becomes more erroneous, and for temperatures below the freezing point the correction even has the wrong sign. It is better, therefore, to use a vapor tension for each degree of temperature which is an average somewhat of tensions observed at that temperature at various places and different seasons of the year. This may be done by substituting in (13) for the two factors $\varphi(P', \tau')$ and $\varphi(P, \tau)$ the expression $[1+f(e)]$ in which, according to Dr. Hann,[26]

$$f(e)=0.00154+.000341\tau$$

This makes the correction for the hygrometric state of the atmosphere a function of the temperature, as the correction introduced by Laplace by means of his modified temperature coefficient, but the latter makes this correction vanish and change signs at 0° C., while by the former it takes place at -4.5° C. Both are therefore somewhat erroneous for low temperatures, since the vapor tension does not entirely vanish at any temperature. It is especially applicable in the cases of averages, but in that of individual observations it is, of course, generally less accurate.

Table VIII is so constructed as to give $\log[1+f(e)]$ for each degree of the argument $(\tau'+\tau)$, which is the sum of the temperatures of the station and of sea-level, but a little allowance is made for the defect in the expression, and the table is so constructed as to not make the correction entirely vanish at any low temperature. With this table, instead of Table II and Table III, we get instead of (15)

16 $$\log \text{R}=\log \text{H}-(\text{Table I}+\text{Table VIII} \\ +\text{Table IV}+\text{Table V}+\text{Table VI})$$

The last two tables here may be omitted, with little error, in middle latitudes and at low altitudes.

313. It is often important to know how much a given amount of change, error, or uncertainty in any of the data entering into the formula of reduction of the barometric pressure to sea-level affects the reduction. Denoting a finite but very small differential of each of the variable quantities by the usual sign of $\varDelta$ prefixed, we get from (13) by differentiation, and using Laplace's vapor correction and neglecting other insensible quantities,

$$\frac{\varDelta P'}{P'}-\frac{\varDelta B_1}{B_1}=\frac{\varDelta H-.002H\varDelta(\tau'+\tau)}{18446.6M[1+.002(\tau'+\tau)]}$$

in which H must be expressed in meters.

This gives, putting $P'=760^{mm}$, as it always is very nearly,

$$(17)\quad \varDelta P'=\frac{760}{B_1}\varDelta B_1+0.095\frac{\varDelta H}{[1+.002(\tau'+\tau)]}-0.000190\frac{H\varDelta(\tau'+\tau)}{[1+.002(\tau'+\tau)]}$$

The first term expresses the effect of any error or uncertainty, $\varDelta B_1$, in the barometric reading of the station, the second, that of an error or uncertainty, $\varDelta H$, in the altitude of the station, and the last, the effect of any error or uncertainty in the temperature used in the reduction, which, we have seen, arises mostly from using the observed temperature at the upper station for the true open-air temperature generally of that altitude, and from errors or uncertainties in reducing this to sea-level for the temperature there.

According to the first term the higher the station, that is, the greater the value of P' in comparison with B_1, the greater the effect of $\varDelta B_1$, and as the altitude of the station diminishes and B_1 increases the value of $\varDelta P'$ approximates to that of $\varDelta B_1$. At a station, therefore, where P' is twice as great as B_1 the effect is nearly twice as great as it is in the case of small altitudes.

From the second term of (17) it is seen that for each meter of error in the value of H the corresponding error in the reduction is nearly 0.095^{mm} if the mean temperature is zero, but for higher temperatures it is a little less. In general, therefore, it requires an error of about 11 meters in the altitude to affect the reduction to sea-level one millimeter.

The last term of (17) shows that the effect of an increase of temperature is to diminish the reduction to sea-level, and that this effect is in proportion to the altitude, and it is readily seen that the usual errors and uncertainties which have been pointed out, in the temperatures used in the reductions to sea-level, give rise to considerable errors in these reductions where the value of H is large. For instance, if there is an error of 1° C. in the average temperature we have $\varDelta(\tau'+\tau)$ equal 2°, and with this value in the formula, we get at zero temperature with $H=1000^{m}$ the effect equal to -0.38^{mm}. With a higher temperature the effect would be a little less. If, therefore, there should be an uncertainty of 5° in the temperature used, as there may often, and even greater, then there would be a corresponding uncertainty of 3.8^{mm} in the reduction to sea-level, or a little less for average temperatures.

314. For each of the stations of the Signal Service, where reductions to sea-level are required every day, the value of P' can be computed from (13), or from the more practical forms which follow it, for the normal annual values of the variables entering into the formulæ. Denoting this constant part of the pressure reduced to sea-level by P'_0, we shall then have

$$P'=P'_0+\Delta P' \tag{18}$$

in which, by (17),

$$\Delta P'=\frac{760}{B_0}\Delta B_1-0.00019\frac{H}{1+.002(\tau'+\tau)}\Delta(\tau'+\tau) \tag{19}$$

In this expression, however, it must be understood that B_0 is the normal value of B_1, and that the finite differentials denote the variations of the variable quantities from their normal annual means. The vapor correction is supposed to be taken in by means of Laplace's modified value of the temperature coefficient, and the variations of humidity by (17) only so far as they depend upon the variations of temperature, the greater the temperature the greater, in general, being the amount of humidity. Each term in the expression of (19) can be given in a small table for such a range of the respective arguments ΔB_1 and $\Delta(\tau'+\tau)$ as will most probably take in the extremes of the deviations. Or since B_0 and the normal temperature of the station are known, the arguments may be the absolute values of B_1 and $(\tau'+\tau)$ included within a range sufficiently large to include extremes.

But a better arrangement would be to use the monthly normals, at least for the last term of (19), and to give the values of $\Delta P'$ in tables, so far as they depend upon each of these terms, for each month of the year. The corrections then to be applied for the deviations from the normals, especially in the case of the last term of (19), would be much smaller, since $\Delta(\tau'+\tau)$ in this case would not be affected by the annual inequality, which at some stations may be quite large, and it would embrace only the abnormal variations during the month.

It is not important that the constant B_0 and the normal average temperature of the upper station and of sea-level $(\tau_0+\tau'_0)$, shall be precisely the true normal values. They may be any assumed constants not differing much from these normals, and consequently, for the sake of convenience, they may often be taken to the nearest unit only, and at least they need not take in all the decimal places usually required.

For English units of measure (19) becomes

$$\Delta P'=\frac{30}{B_0}\Delta B_1-0.000001271\frac{H}{1+.00111(\tau'+\tau-64^\circ)}\Delta(\tau'+\tau) \tag{20}$$

in which H must be expressed in feet.

For any given station B_0 and H are known, and the expression becomes still more simple. For instance, if in the case of Mount Washington we assume $B_0=23.6$ in. and $(\tau_0+\tau_0)=70^\circ$, which differ but little from the normal values, putting $\tau'_0=45^\circ$ the average observed at Portland and Burlington, and putting $\tau_0=25^\circ$, and suppose that the barometer is suspended 6,285 feet above sea-level, it becomes

$$(21) \qquad \varDelta P'=1.27\varDelta B_1-0.00800\frac{\varDelta(\tau'+\tau)}{1+.00111\,(\tau'+\tau-64^\circ)}$$

Each of the terms in this expression for certain values of the variables B_1 and $\tau'+\tau$ within a range on each side of the constant assumed values, so as to take in extreme values, can be given in a table with the arguments respectively B_1 and $(\tau'+\tau)$ and applicable to this station only. The value of P'_0 in (20) to which these corrections are applied must be computed from (13), converted to English measures, or (14) and (16), with the assumed values of B_0 and $(\tau'_0+\tau_0)$ whatever these may be.* The numbers in the tables may be increased by certain constants so as to have them all positive, and then the value of P'_0 must be diminished by the sum of these constants.

For reasons already given in § 311, if we use the full range of deviations from the normal temperature, that is, the full values of $\varDelta(\tau'+\tau)$ in (19), we get very unsatisfactory reductions to sea-level for all extremes of temperature, especially where the changes are sudden and rapid, as in diurnal and sudden abnormal changes. Where $\varDelta(\tau'+\tau)$ represents the observed diurnal inequalities of temperature it has too large a negative value in the morning and too large a positive value in the afternoon generally, especially in clear weather, to represent the inequalities of temperature in the open air away from the earth's surface, which are the inequalities needed in the reductions.

For the times of the mean temperature of the year, as spring or fall, and those of the mean temperature of the day, $\varDelta(\tau'+\tau)$ is generally small, unless there are sudden abnormal changes, and hence at such times reductions to sea-level are more satisfactory. Reductions to sea-level at different hours of the day, where the averages are taken for a length of time sufficiently great to eliminate all abnormal irregularities, should give the same results, except so far as they are affected by the small normal inequality of pressure. But to obtain such a result from the observations made at Geneva and St. Bernard it has been found that the deviations of temperature from the mean of the day must be diminished about two-thirds, that is, if the temperature in the morning is, say, 6° less, or in the afternoon 6° greater than the mean temperature, we must use a temperature deviating from the mean only about 2°. And so in the case of all other inequalities, and the shorter the period of the inequality the more should the inequalities be excluded, and the more nearly should the temperature used in the reductions be that of the mean.

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* See example 1, following.

If a formula similar to that of (21) were made for each month of the year with assumed values of the constants B_0 and $\tau'_0+\tau_0$ which are the monthly normal values, or nearly, then the ranges of the arguments B_1 and $(\tau'+\tau)$ would be considerably less, especially of the latter, than where the deviations are taken from the annual average, and the corrections would generally be smaller.

315. The preceding formulæ and the tables are so constructed as to use directly the wet-bulb temperatures instead of the actual amount of vapor tension deduced by means of (11), § 304. Where the vapor tension or relative humidity is given, and not the temperature of the wet-bulb τ_1, the value of $(\tau-\tau_1)$ may be obtained from the usual tables, such as those of Regnault, which give the vapor tension or relative humidity corresponding to given values of τ_1 and $(\tau-\tau_1)$ used as an argument. This, however, would be much more convenient if τ instead of τ_1 were the argument. Then with the observed value of τ that of τ_1 is readily obtained.

In general the vapor tension is observed at the station only, and not at sea-level, so that the tension corresponding to sea-level which should be used in the formula is unknown. Where, therefore, it is attempted to take into account accurately the effect of vapor tension in the reduction, the best that can be done in obtaining the proper tension to be used at sea-level is to deduce it from the well known empirical formula of Dr. Hann, (10) § 25, which, put into a practical form, is

$$(22) \qquad \log p'-\log p=\frac{H^{m}}{6517}=\frac{H^{ft}}{21380}$$

in which H^{m} and H^{ft} denote the difference of altitude expressed in meters and feet respectively. The value of p being observed, the value of p' at the earth's surface is readily obtained.

316. By reversing (13) we get the usual hypsometrical formula, in which Tables I to IX inclusive can be used for English units of measure. In the practical form, to be used with logarithms, it becomes

$$(23) \qquad \log H=\log(\log P'-\log B_1)+\text{Table I}+\text{Table II (bis)} \\ +\text{Table III (bis)}+\text{Table IV}+\text{Table V}+\text{Table VI}$$

in which $\log B_1$ is given by (8). Where no hygrometric observations are made Table VIII must be used instead of Tables II and III.

In the application of this formula there is the same uncertainty with regard to the temperatures which should be used as in the case of reductions to sea-level. And as these are mostly in the extremes of temperatures, where observations are made especially for the determination of altitude, they should be made, if possible, at times of the mean temperature of the year and day, say in April and October, and about 10 or 11 o'clock in the morning. In cloudy weather, however, when there is little change of temperature, they can be made at any time, and such weather is always preferable.

It is evident from a mere inspection of (23) that the pressures uncorrected for latitude can be used instead of the true pressures, for if they were both corrected by (8) the same quantity from Table IV would be applied to each and the difference would be the same, unless the latitudes of the two stations differed very much. In the case of the temperature correction in Table VIII, this would not be the same, since the arguments for the two stations would be different.

Examples.

1. What is the value of P'_0 in (18), corresponding to the preceding assumed values of H=6285 ft., B^0=23.6 in., and $(\tau'+\tau)$=70° in reducing (20) to (21)? See (14) and (16).

With these data we get log R from (16) and then P'_0 from (14), as follows:

Log H	=3.79831		
Table I (Arg. 70°)	=4.78454	R =	0.10292
Table VIII (Arg. 70°)	=0.00110	log B_1 =	1.37291
Table V (Arg. H)	=0.00013		
Table IV (Arg. 44° 16′)	=0.00003	log P_0'=	1.47583
		$P^{0\prime}$=	29.911
	4.78580		
Log R	=9.01251		

2 With the data of the preceding example, together with an observed relative humidity of 0.85 at the station, which is about the general average, what would be the value of P'? (14) and (15.)

In this example we must use Tables II and III instead of Table VIII, as used above. From Table X with the argument $\tau_1=\tau=25°$ we get 0.135 in. for the vapor tension of saturation. Hence we have

$$p=0.135\times 0.85=0.115 \text{ in.}$$

for the actual vapor tension. With this we get from the Guyot's table $\tau-\tau_1=1.1°$, which is the argument of Table III; and $\tau_1=25°-1.1°=23.9°$ is one of the arguments in Table II.

From (22) we get

$$\log p'-\log p=\frac{6285}{21380}=0.2940 \qquad \log p'=9.0607+0.2940=9.3547$$

This gives $p'=0.226$, and with this Guyot's table gives $\tau'-\tau'_1=3°$, which is the argument to be used in Table III for sea-level; and $\tau'_1=45°-3°=42°$ is the argument of Table II for sea-level.

With these arguments we get

Table II, (Arg., $\tau_1=23.9^\circ$ and $P=23.6$ in.)	0.00045
Table II, (Arg., $\tau_1'=42^\circ$ and $P'=30$ in.)	0.00074
Table III, (Arg., $\tau-\tau_1=1.1^\circ$)	—0.00004
Table III, (Arg., $\tau'-\tau'_1=3.0^\circ$)	—0.00011
	0.00104

With 0.00104, instead of 0.00110 from Table VIII, in the preceding example we get $P_0'=29.912$.

From a comparison of the result in this last example with that of the former, it is seen that the correction in this case for the effect of aqueous vapor by means of Table VIII is sensibly the same as where the correction is made with the observed amount by means of (15). The expression of (16) may therefore be used instead of (15) without sensible error. The correction of Table IV in this example is very small.

3 The mean monthly barometric pressure, corrected for temperature only, for April, 1883, at Denver, Colo., latitude 39° 45′ and altitude 5,294 feet, was 24.582 in. and temperature $\tau=45.6^\circ$ F.; what is the reduced pressure corrected for gravity, P', at sea-level, supposing the temperature to increase 1° F. for each 300 feet of decrease of altitude? (14), (16.)

4 With the same data as in the preceding example and average dew-point 28.6°, what is the pressure reduced to sea-level? (14), (15), (22), and Table X?

Harmonic analysis of barometric observations.

317. Barometric pressures, as well as temperatures, may be analyzed by the harmonic method, and the expressions given by this analysis are important for several reasons. Both the annual and diurnal inequalities of barometric pressure depend upon those of the temperature, and where the latter, as the former, are expressed by a series of harmonic terms depending upon the time, it is interesting and important to trace the relations between the amplitudes and epochs of the temperature and pressure inequalities. These are found to differ very much in different parts of the globe and at different altitudes at the same place. Generally over the earth's surface the epochs of the first and principal inequality in the two cases differ about 180°; that is, the maximum of the one occurs very nearly at the time of the minimum of the other. There are a few places, however, where this is reversed, and where the terms of the maxima and minima are very nearly the same. This occurs mostly over large areas of the northern parts of the Atlantic and Pacific Oceans, for reasons already given, § 205, where it is shown that it is the effect of a large permanent cyclone prevailing over these parts mostly during the winter season, but having an annual inequality. The harmonic analysis of the pressures, therefore, at all the places of the earth's surface where observations are made, shows us the extent of the area so affected by these cyclones.

At great altitudes, also, the times of maxima and minima are reversed, for the most part, in lower and middle latitudes from what they are at the earth's surface. While the maxima at the earth's surface usually occur in mid-winter, at no very great altitude it becomes reversed and occurs in mid-summer. On the top of Pike's Peak, we have seen that the maximum is not only observed in summer, but the pressure is much greater than it is in winter. The same is the case on the top of Mount Washington, but the annual inequality is much less. It is, therefore, interesting to have a harmonic analysis of pressures at different altitudes to see at what altitude this reversal takes place.

The principal *diurnal* inequality of barometric pressure, we know, depends upon that of temperature, and the times of maxima and minima are nearly reversed, the least pressures occurring near the times of greatest temperatures, and *vice versa*, so far as these maxima and minima depend upon these principal inequalities as brought out by the analysis. This, analysis, however, shows that as the altitude increases this inequality diminishes and at no very great altitude it becomes reversed, as in the case of the annual inequality.

Besides the diurnal inequality, there is also a semi-diurnal one, which has never been explained. If this ever receives an explanation it must be through the harmonic analysis, by which the amplitudes and epochs of this inequality are determined for a great many places and compared with one another.

Velocity and direction of the wind.

318. Of all kinds of meteorological observations, those of the velocity and direction of the wind depend most upon local circumstances, and are therefore the most indefinite. From the nature of friction the horizontal motions of the atmosphere arising from almost any kind of disturbing forces, as that of the flow of water in rivers, must be comparatively small very near the earth's surface or bottom of the current and rapidly increase with increase of elevation, and this, in the case of the wind, up to an elevation at least above that of the highest anemometers and wind-vanes. Anemometers, therefore, at different heights in the same locality give different velocities, and their indications give little idea of the comparative velocities of the general motion of the atmosphere at different places at a small distance above the earth's surface, where the current may be supposed to be comparatively little affected by local circumstances, unless the height of the exposure is the same for all and is considerable. Near the earth's surface both the velocities and directions are very much affected by local circumstances, and without taking the effects of all these into account they are not comparable. There are few, perhaps, who have not observed, in crossing rivers on bridges, the great difference between the strength of the wind there, when it blows up or down the river, and that over the country generally, even where it is very level. The velocity of the wind, also, as measured on housetops

in cities, is, no doubt, much less than on the top of an isolated house of the same height in the country. But the velocity of the wind in passing over a ridge or mountain range may be much greater at the top, even near the surface, than at the same level generally at a distance from it, for there is a heaping up of the atmosphere on the windward side and an increased gradient formed where the air passes over, and this is necessary to satisfy the condition of continuity, for where there is a diminished sectional area for the atmosphere in passing over the earth's surface, there must be an increased velocity. The indications of both the anemometer and wind-vane are of course of little value in the valleys and low grounds of a hilly country.

319. There are two principal results obtained from the indications of the anemometer and wind-vane, the one, the degree of windiness of a place, and variableness of the wind's direction, important from a climatic point of view; the other, the resultant of all the motions for a month or a year, showing the general motion of the atmosphere over the earth's surface during the time, unaffected by the numerous temporary and abnormal disturbances, and important in the mechanical theory of the general motions of the atmosphere. There is also the relation between velocity and direction of the wind and the centers of cyclones, important in the theory of cyclones.

We have seen that the air at most places is being continually thrown into gyrations, larger or smaller, as cyclones, tornadoes, and very numerous little temporary whirls at almost all times, giving rise to small fluctuations in the velocity and direction of the wind. The velocities arising from all these, taken in all directions, may indicate a great amount of motion in a given time, and yet the atmosphere may not have moved very far in any direction; in fact, the general motion in any direction may be nothing. On the other hand, in the trade-wind region on the ocean, where the wind blows almost constantly with a steady velocity, and with little change of direction, day after day, the resultant motion is very nearly the whole amount of motion, and measures very nearly the amount of windiness.

Anemometers.

320. These may measure either the velocity or the force of the wind. Of the former kind Robinson's cup-anemometer is now almost exclusively used. Let

w=the velocity of the wind;
v=the gyratory linear velocity of the center of the cups;
r=the distance of this center from the vertical axis;
R=the radius of the cups;
$\rho=w:v$, the ratio between the velocity of the wind and that of the centers of the cups.

Dr. Robinson at first determined, without sufficient experiments, that $\rho=3$, in all cases, whatever the lengths of the arms and diameters of the cups, and this erroneous value of ρ has mostly been used in all countries up to the present time in reducing the wind observations of velocity. The Rev. F. W. Stow first discovered from experiments with anemometers of different patterns that they did not nearly agree with one another or with the Kew pattern. And subsequent experiments have shown that ρ is not a constant for all cases, but that it is a function of r and R, and depends likewise upon several other circumstances. The expression of ρ is mostly an empirical one, though Thiesen[27] has given from theoretical considerations the *form* of an approximate expression of the reciprocal of ρ, from which it is seen that ρ is not only a function of r and R, but likewise of the size of the arms and the density of the air, and that it is also different for different velocities.

In the year 1878 both M. Dohrandt[28] of St. Petersburg and Dr. Robinson[29] at Dublin made numerous experiments by means of whirling machines, both of which showed that the ratio $\rho=3$ is erroneous, and that different values are required for different patterns and for different velocities of the same pattern. Both tested the anemometer of the Kew pattern, and their experimental results agreed very closely, both showing that the value of ρ is nearly equal to 3 for wind velocities from 5 to 10 miles an hour, but that for velocities of 25 or 30 miles per hour it was only about 2.3.

Dohrandt, from the results of experiments with a number of anemometers of different patterns, deduced an empirical formula equivalent to the following:

$$(1) \qquad w=a+cv \qquad\qquad c=3.0133-53.7367\frac{R}{r}R+1033.81\left(\frac{R}{r}\right)^2R^2$$

in which the meter and second are the units of measure and in which the value of a is generally about one meter. This gives

$$(2) \qquad \rho=c+\frac{a}{v}$$

and hence ρ is infinitely great for very small velocities, but as the velocities are increased, its value approximates to that of c.

From these empirical expressions with constants experimentally determined, as well as from Thiesen's theoretical expression with constants undetermined, it is seen that the value of ρ depends upon the ratio between R and r, and by (1_1) and (2) it must increase with decreasing values of this ratio and approximate to

$$(3) \qquad \rho=3.0133+\frac{a}{v}$$

and hence for very long arms and small cup, in which the ratio $R:r$ is very small, and for very large velocities we have very nearly $\rho=3$, as first determined by Dr. Robinson.

In the Kew pattern, however, in which the ratio $R:r$ is comparatively large, we have seen that according to the experiments of Dohrandt and Dr. Robinson, the value of ρ, and consequently that of c, is about 2.3.

According to Thiesen's theoretical deductions the value of ρ increases both with the increase of the size of the arms and the density of the air, but the constants in the terms expressing their effects are undetermined. These can only be determined by experiment.

It is seen, both from experiment and theory, that the value of ρ differs with different patterns of anemometers, and with different velocities of the wind and densities of the air, and therefore, that the proper reductions of the indications of the cup-anemometer, in order to get the velocity of the wind, is not so simple a matter as was at first supposed. And as the simple rule of reduction, based upon the assumed constant $\rho=3$ in all cases, has been generally used, the reductions are mostly quite erroneous, especially where anemometers with comparatively large cups and short arms, as in the Kew pattern, have been used.

321. Pressure anemometers indicate the force of the wind upon a unit of surface, as a square foot, exposed normally to its direction. Lind's anémometer consists of a glass siphon containing water in the lower curved half of it, and having one end of the siphon bent at right angles to the general direction of the wind so as to present a horizontal opening to the action of the wind. Since the pressure of the air arising from this action is the same at all points and in all directions, the force upon a given unit of surface of the water on that side of the siphon is the same as upon the same unit of sectional area in any other part of the tube, and also at the mouth of the tube. The pressure on the surface of the water causes it to descend, and that of the other side to rise until the pressure of the column on that side which is above the level of the surface on the other side is exactly equal to the pressure of the wind. It is not necessary that the tube shall have the same diameter at both surfaces and at the mouth, for the pressure in any case at all parts is the same for a unit of surface or sectional area. The bend of the siphon is, therefore, contracted internally to diminish the jumping movement of the water produced by sudden gusts of wind.

The difference of level between the surfaces of the two sides of the siphon, measured by means of a scale on each side, is a measure of the force of the wind, and as the weight of a cubic foot of water at standard temperature of 62° F. is 62.32 pounds avoirdupois, each inch of this measured difference indicates a force of 5.2 pounds very nearly upon a square foot of surface.

In Osler's anemometer there is a pressure plate one foot square exposed normally to the direction of the wind. This pressure is resisted by springs, and after a certain amount of yielding to the pressure the resistance becomes equal to the pressure. The force belonging to given amounts of yielding of the plate, indicated by a scale, is readily determined and marked upon a scale, and this then indicates the force of the wind.

322. By means of the table, § 232, the forces, as measured by pressure anemometers, can be converted into velocities, or the indications of velocity anemometers, into forces. These relations, however, are based upon theoretical considerations in which no account is taken of the friction of the air, where the wind acts upon a body, as the open end of the tube of Lind's anemometer, or the pressure plate of Osler's anemometer. This body does not simply resist the force of the air of the same sectional area, moving directly toward it, but likewise the action by means of friction of the air on all sides of this in passing by with greater velocity. The effect of this is to increase the pressure on the pressure plate or the mouth of the siphon tube, above what it would be if there were no friction of this kind. The effect of this friction also tends to drag away the air from the other side of the body and to diminish the pressure there, which has the same effect as if so much more force were applied on the other side, for the effective force is the difference of pressure on the two sides. The same takes place in Lind's anemometer, for although there is a vertical partition between the two ends of the siphon, yet the pressure is increased on the one side of this partition and diminished on the other, so that not only the pressure on the one end of the tube is increased by friction, but the pressure at the other end is likewise diminished by it, and consequently the difference of levels of the two sides is increased. The diminution of pressure at the one end would likewise occur to some extent if the partition were not there, for the effect of the air in passing over the open tube is to draw the air within out by the action of friction, and consequently to diminish the pressure.

The velocities, therefore, cannot be accurately determined theoretically from the forces, and *vice versa*, by means of (21), or the following table, in § 230, but the theory gives only an approximate value. This is seen by comparing Loomis's results from the experiments of Newton, § 232, with the results given by formula (24), from which it is seen that the actual resistance in that case was about one-tenth greater than the theoretical.

Hagen's empirical formula in the case of the pressure of the wind upon a plate, determined from very accurate experiments made with a whirling machine, is

$$(4) \qquad p=(0.00707+0.0001125u)Fv^2$$

in which

p=the pressure in grams;
u=the periphery of the plate;
F=the surface of the plate;
v=the velocity per second in decimeters.

The barometric pressure in the experiments was 758^{mm}, and the temperature 15° C.

This formula with p expressed in pounds avoirdupois, u and F in feet, and v in miles per hour, becomes, when expressed so as to include variations of pressure and temperature,

$$(5)\qquad p=(0.003064+0.0001479u)\frac{P}{P_0}\,\frac{Fv^2}{1+.003665\tau}$$

in which τ must be expressed in degrees centigrade.

By comparing this with (21), § 230, it is seen that the experimental value is considerably greater than the theoretical value, and the more so the greater the periphery of the plate. The experiments were made with small plates varying from two to six inches square. How nearly the formula would hold for larger plates, remains to be determined. It would probably be nearly correct for reducing the pressures of the Osler anemometer to velocities, and *vice versa.*

By the cup-anemometer the velocity obtained is the average during a certain short interval of time generally, though a whole day may be assumed as this interval, and then the velocity is the average velocity of the day. Where the wind blows in blasts, therefore, the extreme velocities are not given by this anemometer, since the velocity during even a very short interval of time generally falls considerably below the maximum of the blast.

In the pressure anemometer, where there is a continuous observation or a continuous record of the pressure by means of a curve, all the oscillations of pressure of short period, and consequently the maxima, are given, and by means of the preceding formula the maximum velocities are thus obtained. A self-recording pressure anemometer is, therefore, to be preferred where it is desirable to know the extreme pressures and velocities of the wind, but it is inconvenient for obtaining average velocities, independent of the oscillations of short period, where the wind blows in blasts.

Resultant motions of the air.

323. When the wind blows at different times in different directions and with different velocities, all these motions are equivalent to one motion in the same direction, which is called the resultant motion, and this motion divided by the time is the resultant velocity.

Let us put

A_s= the spaces passed over in the different directions;
φ_s= the corresponding angles from the north reckoned around by the east;
M= the sum of all the components of motion from the north;
N= the sum of all the components from the east;
R= the resultant motion;
β= its angle of direction.

We then evidently have

$$R=\sqrt{M^2+N^2}=\frac{M}{\cos\beta}=\frac{N}{\sin\beta}$$

$$\tan\beta=\frac{N}{M}$$

$$M=\Sigma A_s\cos\varphi_s$$

$$N=\Sigma A_s\sin\varphi_s$$

In these expressions M is simply the sum of all the latitudes and N the sum of all the departures.

It is usual to estimate the directions of the wind at best only to the nearest point, generally only to the nearest one-sixteenth part of the circumference, and frequently only to the nearest half quadrant, so that all directions included within a half division on each side are included in these principal directions. Where the directions are estimated to the nearest point, the values of the characteristic s in the preceding expressions are 0, 1, 2, 3 . . . . 31, these denoting the number of points contained in the angles φ_s. Where the divisions are twice as great they are 0, 2, 4 . . . . 30, and when the divisions are half quadrants they are 0, 4, 8, &c. In all cases we have $\varphi_0=0$.

With the values of A_s for the several given directions φ_s, the values of M and N are readily computed by means of a traverse table, and then with these the values of R and β are soon obtained from the first two of the preceding expressions. The value of R divided by the whole time the wind blew is the resultant velocity.

Where the times of observation are equal, A_s can represent the average velocities during these times, and then the resultant velocity is R divided by the number of directions. This case occurs where observations are made hourly, the observed velocity and direction being supposed to be the average during the hour.

If all the observations at a given hour of the day for a month, or any stated period, be classified with regard to their directions, and the spaces passed over in each of these directions be ascertained, then the resultant of all for this hour can be obtained from the preceding expressions. If this were done for each hour of the day, or only for a few hours at equal intervals during the day, we would get the diurnal inequality in the velocities and directions during the month. The resultants, then, of all these velocities and directions would be the resultants for the month. These would be given pretty accurately generally from only two or three observations a day, at hours suitably chosen, especially where the diurnal inequalities are small.

The resultants obtained for each month would indicate the annual inequalities of velocity and direction, just as those for each hour or a few stated hours indicate the diurnal inequalities. Where the former become completely reversed, or nearly, at the opposite seasons of the year, they indicate a monsoon just as in the latter such reversions on islands and sea-coasts indicate land and sea breezes.

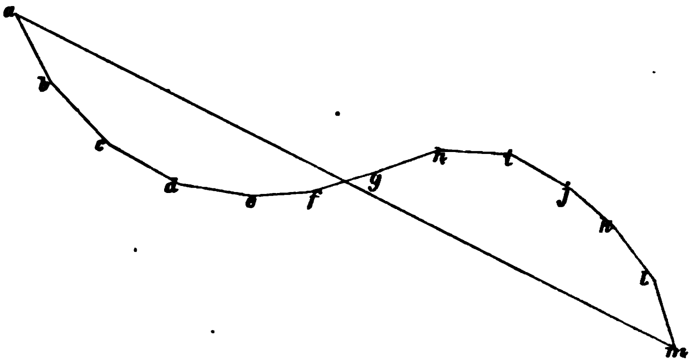

FIG. 1

Where the monthly resultants are found they can be graphically represented, together with the general annual resultant, as in the accompanying figure, in which *ab*, *bc*, *cd*, &c., represent the monthly resultants, commencing with January, and the line *am* the general resultant. As here represented the winds are somewhat northerly at the beginning and end of the year, and a little south of west during the summer, giving a northwesterly annual resultant. If *ab*, *bc*, *cd*, &c., represent mean velocities of the months, then the resultant must be divided by 12 to give the mean annual velocity. If they represent spaces passed over during the month, then *am* represents the resultant of these spaces, or the whole distance passed over in that direction.

Examples.

1. What is the pressure of the wind on a square foot at sea-level and temperature of freezing when the wind has a velocity of 40 miles per hour?

In this example we have $F=1$ and $\tau=0$, and we can put $P=P_0$. Hence we get from (5)

$$p=(0.003064+0.0001479\times4)\times40^2=5.85$$

for the pounds of pressure to the square foot. This, on account of the effect of friction, is considerably more than the theoretical force given by the table of § 230.

2. What would be the pressure for the same velocity at the top of Pike's Peak, with P=18 in. and $\tau=10°$ C.

In this example we have

$$p=5.85\times\frac{18}{30}\times\frac{273'}{283}=3.39.$$

for the pressure in pounds per square foot.

3. If the wind presses with a force of ten pounds on a plate one foot square, with a barometric pressure $P=28$ in. and temperature $\tau=27^\circ$ C., what by (5) is the velocity of the wind per hour?

By reversal we get

$$v=\sqrt{\frac{p\times 30\times 300}{(0.003064+0.0001479\times 4)28\times 273}}=56.8 \text{ miles.}$$

4. During the whole month of July the total distance traveled by the air at Hong-Kong observatory from the different quarters were:

Quarters.	Total distances.
N	76
NE	714
E	5,074
SE	1,478
S	1,001
SW	1,205
W	784
NW	254

What were the resultant distance and direction?

In this example we get from a traverse table:

$$M=76+505-1{,}047-1{,}001-853+180=-2{,}140$$
$$N=505+5{,}074+1{,}047-853-784-180=+4{,}809$$

With these values of M and N the first two expressions of (6) give $\beta=114^\circ$, or the direction $E\ 24^\circ\ S$ and $R=5{,}264$ miles. This value of R divided by the number of hours in July gives 7.07 miles for the average velocity traveled in the direction of the resultant.

5. The following were the velocities v of the wind, in miles per hour and the corresponding values of s, for each hour of the first of July, 1884, at Hong-Kong observatory:

Hours.	v.	s.	Hours.	v.	s.	Hours.	v.	s.
1	18	19	9	7	23	17	3	6
2	18	18	10	6	20	18	5	13
3	19	20	11	10	19	19	6	14
4	25	20	12	13	16	20	7	18
5	18	19	13	12	19	21	9	18
6	11	23	14	10	16	22	10	19
7	8	28	15	3	20	23	14	18
8	5	31	16	2	3	24	7	21

What was the resultant velocity and direction?

6. If we have the following mean velocities and directions for each month of the year, what are the resultant distance and velocity?

Month.	v.	φ	Month.	v.	φ
		°			°
January	18	314	July	12	235
February	16	298	August	14	245
March	17	283	September	16	280
April	15	270	October	16	275
May	14	282	November	17	290
June	11	250	December	16	305

CHAPTER VII.

OCEAN CURRENTS AND THEIR METEOROLOGICAL EFFECTS.

Direct effect of temperature differences.

324. The effect of a difference of temperature in the ocean between the equatorial and the polar regions is very similar to that of the atmosphere. The initial effect, before motion ensues, is to raise the surface of the ocean at the equator a little above the level of the polar regions. The following table is deduced from the results of the temperature soundings of the Challenger Expedition. The temperatures are here given in centigrade degrees, and those for the equator are the means of six soundings.

Depth in fathoms.	Equator.	Lat. 23° 10′ N., long. 38° 42′ W.	Lat. 37° 54′ N., long. 41° 44′ W.
	°	°	°
0	25.5	22.2	21.1
50	17.7	....	....
100	13.1	19.4	17.5
200	8.1	14.8	15.9
300	5.7	11.4	15.6
400	4.6	8.7	12.7
500	3.8	6.5	8.2
600	4.0	5.4	5.8
700	3.9	4.8	4.8
800	3.9	4.1	3.4
900	3.4	4.0	3.2
1,000	2.7	3.5	3.2
1,500	2.3	2.6	2.8
2,500	1.8	....	....

From the temperatures at the equator, given here only in part, Mr. Croll, by means of Muncke's table of the expansion of sea-water, computed the upward expansion of the sea and found it at the equator to be 4.5 feet. The initial effect of this would be to cause a very small surface gradient between the equator and the poles, from which a surface current would flow toward the poles. But in order that the condition of continuity may be satisfied, after the interchanging currents become established, as much water must flow toward the equator in the lower strata of the ocean as flows from the equator in the upper strata, and in order to overcome the friction of this under flow there must be a gradient of decreasing pressure from the polar regions toward the equator. The first surface flow toward the poles, therefore, raises the level a little in the polar regions, and thus gives rise to this pressure gradient in the lower strata, which is greatest at the bottom, and gradually decreases up to some intermediate stratum where it changes sign, and above this the pressure gradient is reversed, since the polar regions are not entirely filled up to the level of the equator, so as to completely destroy the surface gradient of pressure decreasing toward the poles and arising from the upward expansion in the equatorial regions. Instead of a difference of level between the equator and the polar regions of 4.5 feet, it is now perhaps less than two feet, for the greater part of the force arising from the gradients is required for the lower strata, for the friction in them is much greater, since they are resisted both by the bottom of the ocean and the counter-current of the upper strata. On account of this greater friction the volume of water below in motion toward the equator is much greater than that of the upper strata moving from the equator, but of course the velocity is proportionately less. The neutral stratum, therefore, is not midway between the bottom and surface, but much nearer the latter. This has also been proved by the results of the temperature soundings in the North Atlantic, for at a depth which is small in comparison with the whole depth, there is found a great and sudden diminution of temperature, which indicates that the neutral plane is at that depth, and that above that the warm waters flow toward the pole and below it the colder polar water flows toward the equator. There is, therefore, an interchanging motion of the water between the equatorial and polar regions, just as in the case of the atmosphere. The effect of this motion upon the relative temperatures of the equatorial and polar regions, we have seen, is very great.

325. The mobility of the deep water of the ocean is so great that a change of level of only one inch between America and Europe or the equator and the pole would be almost instantly followed by a corresponding motion and change of level. This is known from the action of the moon and sun in producing the tides. The very smallest terms in the development of the lunar and solar tidal forces have their corresponding components in the tidal motions of the ocean, as is well shown from numerous harmonic analyses of tide observations in various parts of the globe. Even in the comparatively shallow sheet of water in Lake Michigan there are regular semi-diurnal oscillations of the water between the northern and southern ends, depending upon the slight semi-diurnal changes of level in that direction due to the varying positions of the moon with regard to the meridian, thus giving rise to a small semi-diurnal tide at Chicago with an amplitude less than an inch. A very small change of level in the sea, therefore, from any cause, is at once followed by a motion of the sea tending to restore it to the original level corresponding to an equilibrium of the forces, and where the

nature of the gradients is such, as between the equator and the poles, as to cause an interchanging motion, this motion being once established, the only forces required are those necessary to overcome the frictional resistances to these motions.

Effect of the earth's rotation.

326. If the whole surface of the earth were covered by the ocean, the effect of the flow of the upper strata toward the poles, on account of the influence of the earth's rotation, would be to cause an easterly component of motion all around the globe, just as in the case of the atmosphere, but this motion could not be so great that the deflecting force toward the equator arising from this component would entirely counteract the force arising from the gradient by which the upper strata are moved toward the poles. This easterly velocity, therefore, as in the case of the upper strata of the atmosphere, has a limit beyond which it cannot go.

Where this easterly flow is interrupted by continents there is a damming up of the water and a flowing around to the right and left, and also a counter-flowing back in the lower strata near the bottom. Thus in the North Atlantic Ocean the surface flow from the equatorial to the polar regions causes, on account of the effect of the earth's rotation, a pressure and a current in the upper strata of the middle latitudes from the coast of America towards Europe. This is turned by the continent partly around to the right between the Azores and the coast of Africa, into the lower latitudes, in part to the left, along the coast of Norway and around by Spitzbergen toward the coast of Greenland, and a part also passes directly back to the American coast in the lower strata. But this deflecting force eastward not only causes an accumulation and a raising of the surface above the general level on the coast of Europe, but it likewise causes a depression of the surface, a partial vacuum, on the side toward America. The surface level of the Gulf of Mexico is a foot or more above that along the American coast in middle latitudes, on account of the general surface gradient between the equator and the poles, and this is increased by the depression arising from the general easterly tendency of the water as explained above, so that for these two reasons the water is drawn from the Gulf of Mexico into the depression about the banks of Newfoundland, and the current deflected around by the coast of Africa passes over in the trade-wind latitudes to supply its place. In a similar manner the water is drawn down from the east coast of Greenland and out of Baffin's Bay into this depression and the current deflected around by the coast of Norway and by Spitzbergen comes in to supply its place. Thus there are two gyrations of the water in the North Atlantic, the one around the central region of the Sargasso Sea, and the other around some point in the northern part of the Atlantic not very far from Iceland.

On account of the peculiar configuration of the coast of Florida and the chain of the West India Islands, the water drawn from the Gulf of Mexico has to pass through the narrow strait of Florida, so that it is here concentrated into a comparatively very narrow and rapid current, called the Gulf Stream, just as in the case of the gentle current of a wide and deep river, where the whole volume has to pass through narrow spaces between islands. A considerable part, however, of the water which comes across from the coast of Africa passes around in a gentle flow on the right-hand side of the West India Islands by the Bahama Islands to join the general eastward current across the Atlantic in middle latitudes.

The warm water flowing from the Gulf tends to the right away from the American coast, for well-known reasons, and the cold stream from the coast of Greenland, called the Greenland current, for the same reasons, tends to the right, and consequently presses toward the American coast. Hence this current hugs the coast and comes in between it and the Gulf Stream, and as warm and cold waters do not mingle freely they are kept separate, so that the dividing nearly vertical plane between is called the cold wall.

The cold water along the American coast does not all come down from higher latitudes, but some of it comes from the under flow from Europe toward America. This under flow on arriving at the American coast tends to gradually rise up toward the surface, and the concentrated warm Gulf Stream is continually cutting a channel through it, so that the cold water only comes up to the surface between the Gulf Stream and the American coast.

327. From the expression of (17) § 145, for computing the gradient of level due to the deflecting force of the earth's rotation, if there was a flow of the upper strata of the North Atlantic from the equator toward the pole with a velocity of only 4 miles in 24 hours, it would give rise to an east and west gradient of sea-level which would cause the surface level on the coast of Europe to be about 10 feet higher than on the American coast, in case of a static equilibrium. But of course the actual difference would be considerably less, since the water, as already stated, would flow away to the right and left, and also back toward the American coast underneath. But it is seen from this what a very small northerly motion of the upper strata is sufficient to give rise to a deflecting force eastward sufficient to cause the easterly motion in the middle latitudes of the North Atlantic, and the accumulation of water on the coast of Europe and the depression on the American coast, and the two gyrations already described. It is also seen how futile it is to attempt to give a rational and satisfactory theory of ocean currents without taking into account the deflecting forces arising from the earth's rotation. It is just as much so as it has been to give a satisfactory explanation of the general motions of the atmosphere, of cyclones, tornadoes, &c., without taking this force into consideration.

328. We have seen that there is a gyration of the water in the Atlantic around the Sargasso Sea. The deflecting force arising from the earth's rotation, being always to the right of the direction of current in the northern hemisphere, drives the surface water imperceptibly from all sides toward the central part of this sea, which is the cause of the collection and retention of great quantities of sea-weed in this sea. As the water tends from all sides toward the central part, of course there is a slight elevation of the sea-level there, and a gradual sinking down of the water in that region and a flowing out below in all directions. This is not only very evident from theoretical considerations, but it is shown by the last two columns of the preceding table, taken from the results of the temperature soundings of the Challenger Expedition. From these it is seen that the temperatures in latitude 23°, and still more in latitude 38°, from a little below the surface all the way down to the bottom, are considerably greater than at the same depths at the equator, showing conclusively that there is a gradual settling down of the warmer surface water of the Sargasso Sea toward the bottom, and consequently a flowing out beneath.

A system of currents similar to that of the North Atlantic is found in the North Pacific, and to some extent in the South Atlantic, South Pacific, and Indian Oceans, especially the gyrations around a central point on the latitude of about 35°. The Japan current, similar to that of the Gulf Stream, except not so narrow and rapid at any point, and the cold current coming in between it and the coast from higher latitudes, are explained in the same manner as those of the North Atlantic. In the South Atlantic there is a warm current passing along the South American coast southward, and a cold one flowing along the western coast of Africa northward, forming parts of a circulation similar to that around the Sargasso Sea. The same holds, to some extent, in the South Pacific and Indian Oceans.

Effects of ocean currents on climate.

329. The effect of these gyrations upon the temperature is to increase it on the side where the motion is from the equator and to decrease it on the other side where it is from the polar regions. Hence the isotherms of these oceans, as has been shown by Dana[30], do not extend east or west but obliquely across these oceans, being farthest from the equator on the sides where the currents are from the equator toward the poles.

As the tendency of the currents, not only of the ocean, but likewise of the atmosphere, in the higher latitudes is from west to east, the effect of the general surface flow of the oceans from the equator toward the poles, and also of the comparatively rapid currents, as the Gulf Stream and Japan current, on the west sides of the oceans, is felt mostly on the eastern sides and the adjacent countries, so that the British Islands have a much higher temperature than Labrador on the same latitudes, and Oregon and Washington Territory than the coast of China on the same latitudes.

The cold currents on the eastern sides of the oceans, as that from the coast of Europe down by the African coast and the Humboldt current along the west coast of South America, and the corresponding one along the west coast of Africa, have a considerable effect upon the temperature of the adjacent coasts, and likewise the Greenland current and the similar one along the coast of China. On account of the great differences of temperature in a short distance between these latter currents and the warmer currents behind which they flow, these cold currents have a dense fog over them whenever the wind carries the warm, moist air of the warm currents over their cold surfaces. This is especially the case in the region of the banks of Newfoundland.

Influence of winds on ocean currents.

330. It was once thought, and still by some, that the winds are the principal, if not the only, cause of ocean currents, and that even the Gulf Stream is due to them. The theory was that the trade winds drive the water of the ocean into the Caribbean Sea and Gulf of Mexico, causing a higher level, and that the Gulf Stream is a current flowing through the strait of Florida from this higher level. But the levelings across the isthmus along the line of the proposed Nicaraguan ship canal has shown that there is no difference of level between the oceans on the two sides, and consequently that the winds have no perceptible influence of that kind. For if they had, they would raise the level on the east side of the isthmus and depress it on the west side and cause a difference of level which would be shown by the levelings.

That the winds, also, have no sensible effect of that sort is shown by tidal observations on the American and European coasts. The analyses of these observations show that sea-level on the European coast is lowest during the winter season, at the very time when the west winds across the Atlantic are the strongest, the velocity at this season being more than twice as great as in the summer season. Furthermore, it is shown from observations of the level at both ends of Lake Ontario, that northeast, east, and southeast winds raise the level at the west end above, and depress it at the east end below, the mean level about one-third of an inch, and that the contrary effect takes place with northwest, west, and southwest winds. The ordinary winds, therefore, produce no sensible effect on sea-level in general, and it is only observable in case of great storms where the whole force is brought to bear upon shallow water, or the water is driven into some bay or gulf which becomes narrower toward the head.

The Gulf Stream, therefore, is not caused by the trade winds raising the mean level of the Gulf of Mexico. It is caused in part by this level being raised by the upward expansion of the warmer water of the Gulf, but in a greater measure by the depression, as explained, in the middle latitudes adjacent to the New England coast. From these two causes there is a descending gradient from the Gulf to this region. This has been recently shown by levelings made in the Coast and Geodetic Survey connecting New York harbor with levelings which had been previously made from the Gulf along the Mississippi River, from which it appears that the level of the Gulf of Mexico is about one meter higher than the Atlantic adjacent to New York.

It must be admitted, however, that the winds, where they blow for some time in the same direction, give rise to surface currents with a considerable velocity, but the Gulf Stream is not due to this cause, for at its origin it flows contrary to the direction of the wind, the wind in the strait of Florida being mostly from the northeast. But when the effect of the earth's rotation upon ocean currents is not taken into account, there is no other satisfactory theory, and then the wind theory seems to be the only one left to account for the ocean currents.

APPENDIX.

HYPSOMETRICAL AND OTHER TABLES.

Table I.—*Containing* log 60518 + log $[1+.001017\ (\tau'+\tau-64^\circ)]$; *Argument* $(\tau'+\tau)$

$\tau'+\tau$	Log.	$\tau'+\tau$	Log.	$\tau'+\tau$	Log.	$\tau'+\tau$	Log.	$\tau'+\tau$	Log.	$\tau'+\tau$	Log.
°		°		°		°		°		°	
0	4.75261	30	4.76659	60	4.78012	90	4.79325	120	4.80600	150	4.81839
1	4.75308	31	4.76705	61	4.78057	91	4.79368	121	4.80642	151	4.81880
2	4.75355	32	4.76750	62	4.78101	92	4.79411	122	4.80684	152	4.81921
3	4.75402	33	4.76796	63	4.78145	93	4.79454	123	4.80726	153	4.81962
4	4.75449	34	4.76842	64	4.78189	94	4.79497	124	4.80768	154	4.82002
5	4.75496	35	4.76888	65	4.78233	95	4.79540	125	4.80810	155	4.82043
6	4.75544	36	4.76933	66	4.78277	96	4.79584	126	4.80851	156	4.82083
7	4.75591	37	4.76979	67	4.78321	97	4.79627	127	4.80893	157	4.82124
8	4.75638	38	4.77024	68	4.78365	98	4.79670	128	4.80935	158	4.82165
9	4.75685	39	4.77169	69	4.78409	99	4.79713	129	4.80977	159	4.82206
10	4.75731	40	4.77114	70	4.78454	100	4.79755	130	4.81018	160	4.82246
11	4.75778	41	4.77160	71	4.78498	101	4.79798	131	4.81059	161	4.82286
12	4.75825	42	4.77205	72	4.78542	102	4.79840	132	4.81101	162	4.82326
13	4.75872	43	4.77250	73	4.78586	103	4.79883	133	4.81143	163	4.82366
14	4.75919	44	4.77295	74	4.78629	104	4.79926	134	4.81184	164	4.82406
15	4.75966	45	4.77340	75	4.78673	105	4.79969	135	4.81225	165	4.82446
16	4.76012	46	4.77385	76	4.78716	106	4.80011	136	4.81266	166	4.82486
17	4.76059	47	4.77430	77	4.78760	107	4.80053	137	4.81307	167	4.82526
18	4.76105	48	4.77475	78	4.78804	108	4.80095	138	4.81348	168	4.82567
19	4.76152	49	4.77520	79	4.78848	109	4.80138	139	4.81389	169	4.82607
20	4.76198	50	4.77565	80	4.78892	110	4.80180	140	4.81430	170	4.82647
21	4.76244	51	4.77610	81	4.78936	111	4.80222	141	4.81472	171	4.82687
22	4.76290	52	4.77655	82	4.78979	112	4.80264	142	4.81513	172	4.82727
23	4.76336	53	4.77000	83	4.79022	113	4.80307	143	4.81554	173	4.82767
24	4.76383	54	4.77745	84	4.79065	114	4.80349	144	4,81595	174	4.82806
25	4.76429	55	4.77790	85	4.79109	115	4.80391	145	4.81636	175	4.82846
26	4.76475	56	4.77834	86	4.79152	116	4.80433	146	4.81676	176	4.82886
27	4.76521	57	4.77879	87	4.79195	117	4.80475	147	4.80717	177	4.82926
28	4.76567	58	4.77924	88	4.79238	118	4.80517	148	4.81758	178	4.82966
29	4.76613	59	4.77968	89	4.79282	119	4.80559	149	4.81799	179	4.83006

Multiples of the differences.

1	47	46	45	44	43	42	41	40	39
2	94	92	90	88	86	84	82	80	78
3	141	138	135	132	129	126	123	120	117
4	188	184	180	176	172	168	164	160	156
5	235	230	225	220	215	210	205	200	195
6	282	276	270	264	258	252	246	240	234
7	329	322	315	308	301	294	287	280	273
8	376	368	360	352	344	336	328	320	312
9	423	414	405	396	387	378	369	360	351

TABLE II.—*Containing* $\log \phi (P, \tau_1)$ *in units of the fifth decimal place; Arguments, P and* τ_1.

P in inches.	τ_1 in degrees Fahrenheit.																		
	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
15	24	30	37	46	58	73	91	111	135	163	197	236	283	337	400	473	557	654	764
16	22	28	35	43	55	68	85	105	127	153	185	222	265	315	373	442	522	614	719
17	21	26	33	41	52	64	80	99	120	145	174	208	249	296	351	416	492	579	678
18	19	24	31	39	49	61	76	93	113	137	164	197	235	280	332	394	466	547	640
19	18	23	30	37	46	58	72	88	107	129	155	187	223	266	316	375	443	520	606
20	17	22	28	35	44	55	68	84	102	123	147	177	212	253	301	357	420	494	575
21	16	21	27	33	42	52	65	80	97	117	140	168	202	242	288	341	400	470	547
22	16	20	26	32	40	50	62	76	92	111	133	161	193	232	275	325	382	448	522
23	15	19	25	30	38	48	59	72	88	106	128	154	184	221	263	311	365	428	499
24	15	18	24	29	36	46	56	69	84	102	123	148	177	211	251	297	350	410	478
25	14	18	23	28	35	44	54	66	81	98	118	142	169	202	241	285	336	394	460
26	14	17	22	27	33	42	52	64	78	94	113	136	162	194	231	274	323	380	444
27	13	17	21	26	32	40	50	62	75	91	109	131	156	187	222	264	311	366	428
28	13	16	20	25	31	39	48	60	72	88	105	126	151	180	214	254	299	352	413
29	12	16	19	24	30	37	46	58	70	85	102	122	145	174	207	245	289	340	398
30	12	15	18	23	29	36	45	56	68	82	99	118	141	168	200	237	280	329	384
31	12	15	17	22	28	35	44	55	66	80	96	115	137	163	194	230	271	318	371

P in millimeters.	τ_1 in centigrade degrees.																
	−18	−15	−12	−9	−6	−3	0	3	6	9	12	15	18	21	24	27	30
500	18	23	29	36	46	59	75	94	116	142	170	208	251	303	361	434	515
520	17	22	28	35	45	57	72	90	111	136	164	200	242	291	348	416	495
540	16	21	27	34	43	56	69	86	106	130	158	193	233	279	336	400	476
560	15	20	26	33	42	54	67	82	102	125	152	186	225	269	324	386	459
580	15	19	25	32	40	52	65	79	98	121	147	179	217	260	313	373	444
600	15	18	24	31	39	50	62	76	95	117	142	173	210	252	303	361	430
620	14	17	24	30	37	48	60	74	92	113	138	167	203	244	293	349	416
640	14	17	23	29	36	46	58	72	89	110	134	162	196	237	284	338	403
660	14	17	22	28	35	45	57	70	87	106	130	157	190	230	275	328	391
680	13	16	21	27	34	44	55	68	84	103	126	152	184	223	267	318	380
700	13	16	21	26	33	43	54	66	82	100	122	148	179	216	259	309	369
720	13	16	20	25	32	41	52	64	79	97	119	144	174	210	252	301	358
740	12	15	19	24	31	40	50	62	77	95	116	140	169	204	246	293	348
760	12	15	19	24	31	39	49	61	75	93	113	137	165	199	240	286	339
780	12	15	18	23	30	38	48	60	73	91	111	134	161	195	235	286	331

TABLE III.—*Containing the complement of log* $[1-.000084\,(\tau-\tau_1)]$; *argument*, $(\tau-\tau_1)$.

$(\tau-\tau_1)$	Complement of logarithm.	$(\tau-\tau_1)$	Complement of logarithm.	$(\tau-\tau_1)$	Complement of logarithm.
° *F.*		° *F.*		° *F.*	
1	−0.00004	11	−0.00041	21	−0.00077
2	.00007	12	.00044	22	.00080
3	.00011	13	.00048	23	.00084
4	.00014	14	.00051	24	.00088
5	.00018	15	.00055	25	.00092
6	.00022	16	.00059	26	.00096
7	.00025	17	.00062	27	.00099
8	.00029	18	.00066	28	.00103
9	.00033	19	.00069	29	.00106
10	.00037	20	.00073	30	.00110

NOTE.—For a well-ventilated psychrometer deduct one-seventh from the logarithms.

TABLE IV.—*Containing the logarithm of* $(1+.002606\cos 2\lambda)$; *argument*, λ.

λ +	Logarithm.	λ −	λ +	Logarithm.	λ −	λ +	Logarithm.	λ −
°		°	°		°	°		°
0	0.00113	90	16	0.00097	74	31	0.00054	59
1	.00113	89	17	.00094	73	32	.00050	58
2	.00113	88	18	.00092	72	33	.00046	57
3	.00113	87	19	.00089	71	34	.00043	56
4	.00113	86	20	.00087	70	35	.00039	55
5	.00112	85	21	.00084	69	36	.00035	54
6	.00112	84	22	.00082	68	37	.00031	53
7	.00111	83	23	.00079	67	38	.00028	52
8	.00110	82	24	.00076	66	39	.00024	51
9	.00108	81	25	.00073	65	40	.00020	50
10	.00107	80	26	.00070	64	41	.00016	49
11	.00105	79	27	.00067	63	42	.00012	48
12	.00104	78	28	.00063	62	43	.00008	47
13	.00102	77	29	.00060	61	44	.00004	46
14	.00101	76	30	.00057	60	45	.00000	45
15	.00099	75						

TABLE V.—*Containing the logarithm of* $\left(1+\frac{H}{r'}\right)$; *argument*, *H*.

H	Logarithm.	*H*	*H*	Logarithm.	*H*
Feet.		*Meters.*	*Feet.*		*Meters.*
100	0.00000	30	2,000	0.00004	610
200	.00001	61	3,000	.00006	914
300	.00001	91	4,000	.00008	1,219
400	.00001	122	5,000	.00010	1,524
500	.00001	152	6,000	.00012	1,829
600	.00001	183	7,000	.00014	2,134
700	.00001	213	8,000	.00016	2,438
800	.00002	244	9,000	.00018	2,743
900	.00002	274	10,000	.00020	3,048
1,000	.00002	305			

TABLE VI.—*Containing the logarithm of* $\left(1+\frac{2h'}{r'}\right)$; *argument, h'.*

h'	Logarithm.	h'	h'	Logarithm.	h'
Feet.		*Meters.*	*Feet.*		*Meters.*
1,000	0.00004	305	6,000	0.00024	1,824
2,000	.00008	610	7,000	.00028	2,134
3,000	.00012	914	8,000	.00032	2,438
4,000	.00016	1,219	9,000	.00036	2,743
5,000	.00020	1,524	10,000	.00039	3,048

TABLE VII.—*Containing the logarithm of* $[1+.0000895\,(\tau-28.5^\circ)]$; *argument,* τ.

τ	Complement of logarithm.	τ	Complement of logarithm.	τ	Complement of logarithm.
0	−0.00111	36	+0.00029	72	+0.00168
3	.00099	39	.00041	75	.00180
6	.00088	42	.00053	78	.00192
9	.00076	45	.00064	81	.00203
12	.00064	48	.00075	84	.00215
15	.00053	51	.00087	87	.00227
18	.00041	54	.00099	90	.00239
21	.00029	57	.00111	93	.00251
24	.00018	60	.00122	96	.00262
27	.00006	63	.00133	99	.00275
30	+0.00006	66	.00145	102	.00287
33	.00017	69	.00156	105	.00299

TABLE VIII.—*To be used in place of Tables II and III where no hygrometric observations are made.*

$\tau'+\tau$	Logarithm.	$\tau'+\tau$	$\tau'+\tau$	Logarithm.	$\tau'+\tau$
° *F.*		° *C.*	° *F.*		° *C.*
0	0.00030	−17.8	100	.00215	+37.8
10	.00037	11.2	110	.00256	43.3
20	.00045	6.7	120	.00297	48.9
30	.00053	1.1	130	.00338	54.4
40	.00062	+ 4.5	140	.00379	60.0
50	.00073	10.0	150	.00420	65.5
60	.00088	15.6	160	.00460	71.1
70	.00110	21.1	170	.00501	76.6
80	.00138	26.7	180	.00542	82.2
90	.00175	32.2			

TABLE IX.—*Gravity correction for a pressure of* 760 *mm., or* 30 *inches at latitude λ; argument, λ.*

λ −	Correction.		λ +	λ −	Correction.		λ +
	mm.	*Inches.*			*mm.*	*Inches.*	
0	1.98	.078	90	23	1.37	.054	67
1	1.98	.078	89	24	1.33	.052	66
2	1.97	.078	88	25	1.27	.050	65
3	1.97	.078	87	26	1.22	.048	64
4	1.96	.077	86	27	1.16	.046	63
5	1.95	.077	85	28	1.11	.044	62
6	1.94	.076	84	29	1.05	.041	61
7	1.92	.076	83	30	0.99	.039	60
8	1.90	.075	82	31	0.93	.037	59
9	1.88	.074	81	32	0.87	.034	58
10	1.86	.073	80	33	0.81	.032	57
11	1.84	.072	79	34	0.74	.029	56
12	1.81	.071	78	35	0.68	.027	55
13	1.78	.070	77	36	0.61	.024	54
14	1.75	.069	76	37	0.55	.022	53
15	1.72	.067	75	38	0.48	.019	52
16	1.68	.066	74	39	0.41	.016	51
17	1.64	.065	73	40	0.34	.014	50
18	1.60	.063	72	41	0.28	.011	49
19	1.56	.062	71	42	0.21	.008	48
20	1.52	.660	70	43	0.14	.005	47
21	1.47	.058	69	44	0.07	.003	46
22	1.42	.056	68	45	0.00	.000	45

TABLE X.—*Containing the tension of aqueous vapor in saturated air, p_1, at the temperature τ_1.*

τ^1	Tension.	τ^1	Tension.	τ^1	Tension.	τ^1	Tension.	τ^1	Tension.
°*F.*	*Inches.*	°*F.*	*Inches.*	°*F.*	*Inches.*	°*F.*	*Inches.*	°*F.*	*Inches.*
0	0.045	20	0.109	40	0.246	60	0.517	80	1.021
1	0.047	21	0.114	41	0.256	61	0.536	81	1.055
2	0.049	22	0.119	42	0.266	62	0.555	82	1.090
3	0.051	23	0.124	43	0.276	63	0.575	83	1.126
4	0.054	24	0.129	44	0.287	64	0.595	84	0.163
5	0.057	25	0.135	45	0.298	65	0.616	85	0.201
6	0.059	26	0.141	46	0.310	66	0.638	86	1.239
7	0.062	27	0.147	47	0.322	67	0.660	87	1.279
8	0.065	28	0.153	48	0.334	68	0.683	88	1.320
9	0.068	29	0.159	49	0.347	69	0.707	89	1.363
10	0.071	30	0.166	50	0.360	70	0.732	90	1.407
11	0.075	31	0.173	51	0.374	71	0.757	91	1.552
12	0.078	32	0.180	52	0.388	72	0.783	92	1.498
13	0.081	33	0.187	53	0.402	73	0.810	93	1.545
14	0.085	34	0.195	54	0.417	74	0.837	94	1.594
15	0.088	35	0.203	55	0.432	75	0.865	95	1.644
16	0.092	36	0.211	56	0.448	76	0.894	96	1.695
17	0.096	37	0.219	57	0.464	77	0.925	97	1.748
18	0.100	38	0.228	58	0.481	78	0.956	98	1.802
19	0.104	39	0.237	59	0.499	79	0.988	99	1.857
°*C.*	*mm.*	°*C.*	*mm.*	°*C.*	*mm.*	°*C.*	*mm.*	°*C.*	*mm.*
−18	1.12	6	2.93	5	6.51	16	13.51	27	26.47
17	1.22	−5	3.16	6	6.97	17	14.40	28	28.07
16	1.32	−4	3.41	7	7.47	18	15.33	26	29.74
15	1.44	3	3.67	8	7.99	19	16.32	30	31.51
14	1.56	2	3.95	9	8.55	20	17.36	31	33.37
13	1.69	−1	4.25	10	9.14	21	18.47	32	35.32
12	1.84	0	4.57	11	9.77	22	19.63	33	37.37
11	1.99	+1	4.91	12	10.43	23	20.86	34	39.52
10	2.15	2	5.27	13	11.14	+24	22.15	35	41.78
9	2.33	3	5.66	14	11.88	25	23.52	36	44.16
8	2.51	4	6.07	15	12.67	26	24.96	+37	46.65
7	2.72								

TABLE XI.—*Containing the values of the intensity of solar radiation J and solar constant A in terms of the mean solar constant A_0.*

Date.	Day of year.	*u*.	Latitudes.										A
			0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	
		°											
Jan. 1..	1	00.99	.303	.265	.220	.169	.117	.066	.018	......	......	......	1.0335
16..	16	15.78	.307	.271	.229	.180	.129	.078	.028	......	......	......	1.0324
Feb. 1..	32	31.54	.312	.282	.244	.200	.150	.100	.048	.006	......	......	1.0288
15..	47	45.84	.317	.293	.261	.223	.177	.118	.075	.027	......	......	1.0235
Mar. 1..	60	59.14	.320	.303	.279	.245	.204	.158	.108	.056	.013	......	1.0173
16..	75	73.93	.321	.313	.296	.270	.236	.195	.148	.097	.057	......	1.0096
Apr. 1..	91	89.70	.317	.319	.312	.295	.269	.235	.195	.148	.101	.082	1.0009
16..	106	104.49	.311	.321	.323	.315	.297	.271	.238	.201	.175	.177	0.9923
May 1..	121	119.29	.303	.318	.330	.329	.320	.302	.278	.253	.255	.259	0.9841
16..	136	134.05	.294	.318	.333	.339	.337	.327	.312	.298	.317	.322	0.9772
June 1..	152	149.82	.287	.315	.334	.345	.349	.345	.337	.344	.360	.366	0.9714
16..	167	164.60	.283	.313	.334	.348	.354	.353	.348	.361	.378	.384	0.9679
July 1..	182	179.39	.283	.312	.333	.347	.352	.351	.345	.356	.373	.379	0.9666
16..	197	194.13	.287	.314	.332	.342	.345	.340	.329	.331	.347	.352	0.9674
Aug. 1..	213	209.94	.294	.316	.330	.334	.330	.318	.300	.282	.295	.300	0.9709
16..	228	224.73	.303	.318	.325	.322	.310	.291	.264	.234	.227	.231	0.9760
Sept. 1..	244	240.50	.310	.318	.316	.305	.285	.256	.220	.180	.139	.140	0.9828
10..	259	255.29	.315	.315	.305	.284	.256	.220	.178	.130	.107	.043	0.9909
Oct. 1..	274	270.07	.317	.308	.289	.261	.225	.183	.135	.084	.065	......	0.9995
16..	289	284.86	.316	.298	.271	.236	.194	.147	.097	.047	.015	......	1.0080
Nov. 1..	305	300.63	.312	.280	.251	.211	.164	.114	.063	.018	......	......	1.0164
16..	320	315.42	.308	.276	.235	.190	.140	.089	.040	......	......	......	1.0235
Dec. 1..	335	330.19	.304	.267	.224	.175	.124	.072	.024	.....	......	.....	1.0288
16..	350	344.98	.302	.263	.218	.167	.115	.064	.016	......	......	......	1.0323
Years..	......	....	.305	.301	.289	.268	.241	.209	.173	.144	.133	.126	....

TABLE XII.—*Containtng the values of* ε *&* $\frac{1}{2}\varepsilon(\varepsilon-1)$ *in* (101), § 50, *and also the corresponding values of Sec. z.*

z.	ε.	$\frac{1}{2}\varepsilon(\varepsilon-1)$	Sec. z.	z.	ε.	$\frac{1}{2}\varepsilon(\varepsilon-1)$	Sec. z.	z.	ε.	$\frac{1}{2}\varepsilon(\varepsilon-1)$	Sec. z.
0	1.000	0.00	1.000	42	1.344	0.23	1.346	67	2.548	1.99	2.559
10	1.015	0.01	1.015	43	1.365	0.25	1.367	68	2.655	2.21	2.668
15	1.035	0.02	1.035	44	1.388	0.27	1.390	69	2.776	2.47	2.791
20	1.064	0.03	1.064	45	1.412	0.29	1.414	70	2.906	2.78	2.924
21	1.071	0.03	1.071	46	1.436	0.31	1.439	71	3.051	3.15	3.072
22	1.078	0.03	1.078	47	1.463	0.33	1.466	72	3.211	3.57	3.236
23	1.086	0.04	1.086	48	1.492	0.36	1.495	73	3.391	4.07	3.421
24	1.095	0.04	1.095	49	1.522	0.39	1.525	74	3.592	4.66	3.628
25	1.103	0.05	1.103	50	1.554	0.42	1.557	75	3.822	5.39	3.864
26	1.112	0.05	1.112	51	1.587	0.46	1.596	76	4.087	6.31	4.133
27	1.121	0.06	1.122	52	1.622	0.50	1.625	77	4.391	7.45	4.445
28	1.132	0.07	1.133	53	1.659	0.55	1.662	78	4.742	8.87	4.809
29	1.143	0.08	1.144	54	1.697	0.60	1.701	79	5.157	10.74	5.241
30	1.154	0.08	1.155	55	1.739	0.65	1.743	80	5.62	12.98	5.76
31	1.166	0.09	1.167	56	1.784	0.70	1.788	81	6.20	16.1	6.39
32	1.178	0.10	1.179	57	1.832	0.76	1.836	82	6.90	20.4	7.18
33	1.192	0.11	1.192	58	1.883	0.83	1.887	83	7.78	26.4	8.21
34	1.205	0.12	1.206	59	1.938	0.91	1.942	84	8.91	39.1	9.57
35	1.220	0.13	1.221	60	1.996	1.00	2.000	85	10.48	49.7	11.47
36	1.235	0.14	1.236	61	2.059	1.10	2.063	86	12.6	. . .	14.3
37	1.251	0.15	1.252	62	2.125	1.20	2.130	87	15.5	. . .	19.1
38	1.247	0.16	1.269	63	2.197	1.33	2.203	88	19.9	. . .	28.7
39	1.285	0.17	1.287	64	2.274	1.47	2.281	89	26.2	. . .	57.3
40	1.303	0.19	1.305	65	2.358	1.62	2.366	90	35.5	. . .	
41	1.323	0.21	1.325	66	2.450	1.79	2.459				

TABLE XIII.—*Containing the diminution of temperature for each* 100 *meters of ascending saturated air.*

Pressure P.	Temperature τ.									*Altitude for 0°.
	−10°.	−5°.	0°.	5°.	10°.	15°.	20°.	25°.	30°.	
Mm.	°	°	°	°	°	°	°	°	°	*Meters.*
760	0.74	0.68	0.64	0.58	0.53	0.48	0.43	0.40	0.37	0
700	.73	.66	.63	.57	.51	.46	.42	.38	.36	660
600	.70	.63	.60	.54	.48	.43	.40	.36	......	1897
500	.66	.60	.56	.50	.45	.40	.37	......	......	3357
400	.62	.55	.51	.46	.41	.37	......	......	......	5142
300	.56	.49	.46	.42	......	......	......	......	......	7550
200	.48	.41	.39	......	......	......	......	......	......	10680

WEIGHT OF AQUEOUS VAPOR IN A KILOGRAM OF SATURATED AIR.

Mm.	*Grams.*	*Grams.*	*Grams.*	*Grams.*	*Grams.*	*Grams.*	*Grams.*	*Grams.*	*Grams.*	*Meters.*
760	1.7	2.6	3.8	5.4	7.6	10.5	14.4	19.5	26.3	0
600	2.2	3.2	4.6	6.8	9.6	13.3	18.3	24.8	....	1897
400	3.3	4.8	7.2	10.2	14.4	20.0	....	....	....	5142
200	6.5	9.7	....	....	....	....	....	....	....	10680

* Computed by (50), § 18, with the assumed temperatures of the table of § 27. These altitudes are some greater for high summer temperatures, and less for winter temperatures.

TABLE XIV.—*Containing the values of constants, and combinations of constants and their logarithms, used in the formulæ of the preceding chapters.*

$G=9.806056$ meters [0. 9914944]
$r=6367323$ meters [6. 8039569]
$n=.00007291$ [0. 86285–5]
$rn=464.31$ meters [2. 66681]
$rn^2=.033858$ [0. 52966–2]
$\frac{rn^2}{G}=\frac{1}{289.62}$ [0. 53817–3]

$l=7991.7$ [3. 90264]
$gl=7.8366$ [4. 89418]
$M=.43429$ [0. 63778–1]
1 mile=1609. 32 meters [3. 20664]
1 kilogram=2. 20485 pounds [0. 343379]
1 meter=3. 2809 feet [0. 5159929]

FACTORS OF REDUCTION.

Meters per second to miles per hour 2. 2370 [0. 34966].
Pounds avoirdupois to pounds Troy 1.215278 [0. 0846756].
Grams per square centimeter to pounds avoirdupois per square foot 2.04830 [0. 311898].
Feet per second to miles per hour 0. 6818182 [0. 8336686–1].

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Erratum.

In Fig. 5, page 206, the line B C should be drawn as a chord within the arc of the circle.

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